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BALLISTICS

PART II

(EXTERIOR BALLISTICS)

Prepared for the Department of Artillery and Gun Defense

by

ALSTON HAMILTON, Captain, Coast Artillery Corps

COAST ARTILLERY SCHOOL PRESS
FORT MONROE, VA.
1908

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Second Section
General Staff,

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PREFACE

In the preparation of this brief outline of accepted methods of dealing with Exterior Ballistics, the works of Euler, Otto, Siacci, Parodi, Ingalls, Ronca, and Charbonnier have been consulted. On account of the finished classical methods current at the present time little remained for the writer to do beyond selecting the best for use in this school. For the appendices the writer assumes entire responsibility. In Appendix I the Report of the Chief of the Weather Bureau, for 1900, and the Annuaire du Bureau des Longitudes (1900) are used as authorities on the temperature gradient.

Appendix II was indebted for its basic hypothesis to the rule used in France of having the characteristic index indentical with the sine of the semi-ogival angle. The courtesy of Colonel Rogers Birnie, Ordnance Department, U. S. Army, and of Commander R. H. Jackson, U. S. Navy, has rendered possible the experimental verification of the formulas deduced.

Appendix III is influenced partly by the writer's own experience and partly by the work of Major Chapel. The values of the function of retardation as determined by the Gâvre Commission were, of course, an indispensable element of the discussion.

ALSTON HAMILTON.

Fort Monroe, Va., September 9, 1908.

TABLE OF CONTENTS

1. The Principal Problem 1
2. The Inverse Problem 4
3. General Character of the Trajectory 5
4. Nomenclature of the Trajectory 8
5. Limiting Cases 10
6. Applications 13
7. Direct Fire 17
    Formulas for the Point of Fall 22
    Formulas for the Summit 22
    Formulas for any Point 22
    Forms Analogous to those in Vacuo 23
8. Case of $F(v) = A_n n^n$ 23
9. Quadratic Law of Resistance 26
    Formulas for Point of Fall and Summit 26
    Formulas for Any Point 27
10. The Integrating Factor 27
11. Differential Formulas: General 28
12. Special Differential Formulas 29

13. Solutions by Quadratures:

    Case of $F(v) = A_n v^n$ 31
    The Quadratic Law 31
14. Appendix I. The Altitude Factor 35
15. Appendix II. The Coefficient of Form 37
16. Appendix III. Experimental Determinations of F(v) 43

ERRATA.

Page 6 line 11: read “which” for “whieh”
“ 6 line 11: read “ι” for “i”
“ 7 line 15: read “ι” for “i”
“ 8 line 12: read “$r = -\infty$” for “$r = \infty$”
“ 8 line 14: insert “,” after “v_o” instead of “.”
“ 8 line 18: “;” after $\frac{\pi}{2}$
“ 16 line 18: read “formula” for “formala”
“ 17 line 32: read “pseudo” for “preudo”
“ 20 line 27: read “$(n - 2)\ A_n$” for “$(n - 1)\ A_n$”
“ 24 line 11: read “$\frac{e^{2m} - 1 - 2\ m}{2\ m^2}$” for “$e^{2m} - 1 - 2\ m$”
“ 45: Values of $K(v) \times 10^4$ should be:
opposite $v = 1214$, 1.1824 instead of 1.1784
opposite $v = 1378$, 1.3240 instead of 1.3213
opposite $v = 1607$, 1.3591 instead of 1.3561
“ 45 last line: read “1.0308” for “10.308”

ANALYTICAL
EXTERIOR BALLISTICS.

Satisfactory analytical treatment of the flight of projectiles requires the separate discussion of the *principal problem*, as distinguished from the various *secondary problems* necessary for a complete practical solution.

The principal problem is solved by using certain hypotheses which are broadly true and which permit of a discussion of the problem without the restraint that would ensue if all the less significant features of the problem were introduced.

The secondary problems are those in which are determined the effect of the various circumstances omitted in the discussion of the principal problem, in order that proper compensation may be made.

In Part I these effects have been given and also the method of compensation in each case. The discussion of the secondary problems has been in some instances omitted from Part I, to be included in Part II.*

In Part I. the principal problem was not solved, but the tables resulting from its solution in the cases of practical importance were assumed and used in the solution of ordinary problems.

In Part II. the principal problem is first discussed in its most general form and the method of solution outlined.

Particular hypotheses are then made permitting of solution of the problem for direct, curved, and high angle fire, and the calculation of tables for use in the solution of ordinary problems.

The Principal Problem.

The principal problem in exterior ballistics considers the path and circumstances of motion of a projectile moving with a velocity v, at an inclination θ with the horizontal, the retardation being expressed by $\frac{F(v)}{C}$, in which F(v) is an increasing function of v, and C (see Chapter II, Part I) an increasing function of the ability of the projectile to overcome the resistance of the air. Gravity is supposed constant in intensity and acting vertically downward and the earth plane and motionless. The air is supposed to be of uniform density.

---

* See Appendix I, Part II.

Under these conditions

$$\frac{dv}{dt} = -\frac{F(v)}{C} \tag{1}$$

Taking as an origin some particular point on the path of the projectile for which values of v and θ will, for purposes of distinction, be represented by V and φ respectively; and representing distances measured horizontally in the direction of motion by x and those measured in an upward vertical direction by y, we have, at any time, t seconds after the projectile has left the origin, for the horizontal and vertical components, respectively, of the acceleration

$$\frac{d^2x}{dt^2} = \frac{d(v\cos\theta)}{dt} = -\frac{F(v)}{C}\cos\theta \tag{2}$$

$$\frac{d^2y}{dt^2} = \frac{d(v\sin\theta)}{dt} = -\frac{F(v)}{C}\sin\theta - g \tag{3}$$

or

$$\cos\theta\frac{dv}{dt} - v\sin\theta\frac{d\theta}{dt} = -\frac{F(v)}{C}\cos\theta \tag{2'}$$

$$\sin\theta\frac{dv}{dt} + v\cos\theta\frac{d\theta}{dt} = -\frac{F(v)}{C}\sin\theta - g \tag{3'}$$

From (2′) and (3′) we find by combination

$$\frac{dv}{dt} + \frac{F(v)}{C} = v\tan\theta\frac{d\theta}{dt} = -v\cot\theta\frac{d\theta}{dt} - \frac{g}{\sin\theta} \tag{4}$$

whence

$$\frac{vd\theta}{dt} = -g\cos\theta \tag{5}$$

From (2 and (5), on eliminating dt

$$\frac{d(v\cos\theta)}{d\theta} = \frac{vF(v)}{gC} \tag{6}$$

The accelerations involved in equations (1), (2), (3) and (5) are illustrated in Figure 1.

In the figure

$PA = -g$; $PB = -\frac{F(v)}{C}$; PC the resultant of PA and PB.

$$PD = PB\cos\theta = \frac{d(v\cos\theta)}{dt} = \frac{d^2x}{dt^2} = -\frac{F(v)}{C}\cos\theta$$

$$PE = \frac{d(v\sin\theta)}{dt} = \frac{d^2y}{dt^2} = -g - \frac{F(v)}{C}\sin\theta$$

$$PF = \frac{dv}{dt}$$

$$PG = \frac{vd\theta}{dt} = -g\cos\theta$$

From equation (5)

$$dt = -\frac{v}{g} \sec\theta \, d\theta \qquad (7)$$

and since $dx = v \cos\theta \, dt$; $dy = v \sin\theta \, dt$; $ds = vdt$,

$$dx = -\frac{v^2}{g} \, d\theta \qquad (8)$$

$$dy = -\frac{v^2}{g} \tan\theta \, d\theta \qquad (9)$$

$$ds = -\frac{v^2}{g} \sec\theta \, d\theta \qquad (10)$$

Of the equations (6), (7), (8), (9), (10) there is but one which introduces the law of retardation and this is equation (6), which, as it connects the velocity and its direction is evidently the equation of the hodograph and will be referred to as such.

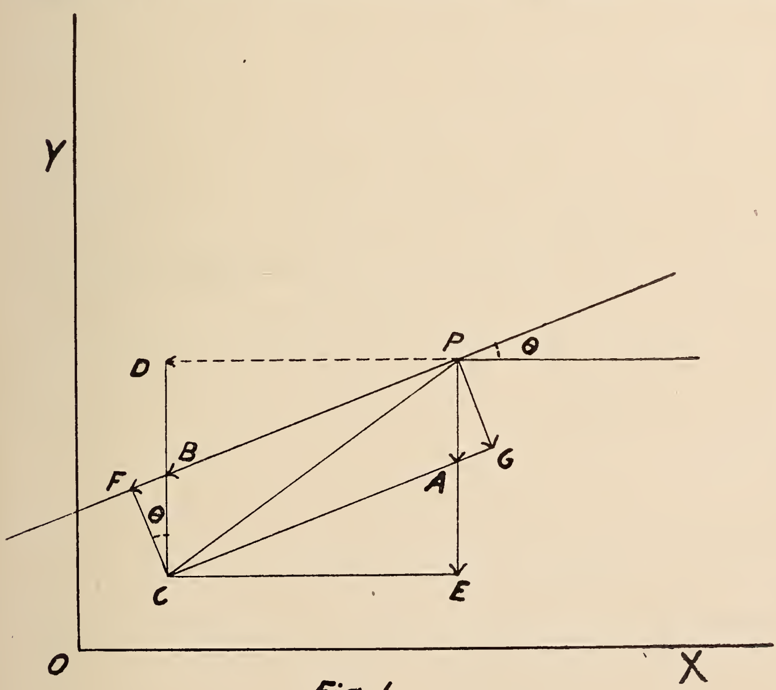


Fig. 1.

The integration of the equation of the hodograph, permits the immediate reduction to quadratures of the complete solution of the principal problem. For, if v may be expressed as a function of θ, equations (7), (8), (9) and (10) will on the substitution of this value of v, express dt, dx, dy and ds in terms of θ and $d\theta$. Equations (7), (8), (9), (10) are general, and the equation of the hodograph furnishes the data for solving the direct problem.

The Inverse Problem.

By means of the equation of the hodograph and the relation that it affords between v and θ, it is clear that x, y, t and s may each be, theoretically at least, expressed as a function of θ alone; and, by the elimination of θ between two of the resulting equations, any one element may be expressed as a function of any other.

If, on the other hand experimental firings show that a relation exists between any two of these elements, this relation, by the solution of an inverse problem, makes possible the detection of the law of retardation and the other elements. Suppose, for instance, that $f(x,y) = 0$ is found experimentally to represent the equation of the trajectory. It is desired to find $\frac{F(v)}{C}$ and the other elements.

1 $$\tan\theta = \frac{dy}{dx} = y'$$

2 $$\frac{d\tan\theta}{dx} = \frac{\sec^2\theta\, d\theta}{dx} = \frac{d^2y}{dx^2} = y''$$

3 $\frac{d\theta}{dx} = -\frac{g}{v^2}$ and $\sec^2\theta = 1 + y'^2$, and, hence, from (2)

$$-\frac{g}{v^2} = \frac{y''}{1+y'^2} \text{ or } v^2 = -\frac{g\,(1+y'^2)}{y''}, \text{ and } v^2\cos^2\theta = -\frac{g}{y''}$$

4 $$\frac{dx}{dt} = v\cos\theta, \text{ hence } \frac{dt^2}{dx^2} = -\frac{y''}{g}$$

Hence

$$dt = \sqrt{-\frac{y''}{g}}\, dx$$

$$\text{and } t = \int_0^x \sqrt{-\frac{y''}{g}}\, dx$$

5 Since $\frac{dx^2}{dt^2} = -\frac{y''}{g}$, differentation with respect to t gives

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -\frac{gy'''}{y''^2}\cdot\frac{dx}{dt}$$

in which $y''' = \frac{d^3y}{dx^3}$

Hence $\frac{d^2x}{dt^2} = -\frac{gy'''}{2y''^2} = -\frac{F(v)}{C}\cos\theta = -\frac{F(v)}{C\sec\theta}$

and from this

$$\frac{F(v)}{C} = -\frac{g}{2}\cdot\frac{y'''}{y''^2}\cdot\sqrt{1+y'^2} = -\frac{y'''}{2g}\cdot v^4\cos^3\theta$$

As a general illustration, suppose that, assuming an equation

$$y = x \tan \varphi - \frac{gx^2}{2\,V^2 \cos^2\varphi} \cdot Q$$

to represent the relation between y and x, and that it is found by experiment that

$$Q = 1 + \psi\left(V, \frac{x}{C}\right)$$

this being a relation obtained from the plotting of the observed values of V and X for y = 0, (*i. e.* at the point of fall) so that

$$\frac{V^2 \sin 2\varphi}{gX} = 1 + \psi\left(V, \frac{X}{C}\right).$$

Then, representing the successive derivatives of Q with respect to x by Q′, Q″, Q‴, etc., we have

$$y' = \frac{dy}{dx} = \tan\varphi - \frac{g}{V^2 \cos^2\varphi}\left(x\,Q + \frac{x^2}{2} \cdot Q'\right)$$

$$y'' = \frac{d^2y}{dx^2} = - \frac{g}{V^2 \cos^2\varphi}\left(Q + 2\,x\,Q' + \frac{x^2}{2}\,Q''\right).$$

$$y''' = - \frac{g}{V^2 \cos^2\varphi}\left(3\,Q' + 3\,x\,Q'' + \frac{x^2}{2}\,Q'''\right)$$

and hence

$$\frac{F(v)}{C} = - \frac{y'''}{2g} \cdot v^4 \cos^3\theta = \frac{3\,Q' + 3\,x\,Q'' + \frac{1}{2}\,x^2\,Q'''}{2\,V^2 \cos^2\varphi} \cdot v^4 \cos^3\theta$$

whence the law of retardation is known.

Since for $V = 0$ or $\frac{x}{C} = 0$, $Q = 1$, (we then have the conditions in vacuo,) it follows that $\psi\left(V, \frac{x}{C}\right)$ must vanish with V, or vanish with $\frac{x}{C}$. Hence $\psi\left(0, \frac{x}{C}\right) = 0$ and $\psi(V, 0) = 0$.

The solution of the inverse problem from observations in curved and high angle fire is particularly useful as throwing light on the laws that govern the flight of oblong projectiles at considerable elevations, which, as experimental results indicate, differ considerably from the laws ordinarily taken as a basis; and this difference seems to be due to the behaviour of the projectile in flight, which, while systematic, is such that the resistance is not a function of the velocity alone.

General Character of the Trajectory.

Resuming the differential equations of motion

$$\frac{d\,(v \cos\theta)}{d\theta} = \frac{v\,F(v)}{g\,C}$$

$$dt = -\frac{v}{g} \cdot \sec\theta \, d\theta$$

$$dx = -\frac{v^2}{g} d\theta$$

$$dy = -\frac{v^2}{g} \tan\theta \, d\theta$$

$$ds = -\frac{v^2}{g} \sec\theta \, d\theta$$

1. Let it be required to trace the change in velocity in the trajectory from the equation of the hodograph, the latter being placed in the form

$$\frac{dv}{d\theta} = v \sec\theta \left(\frac{F(v)}{gC} + \sin\theta\right).$$

a. When v is infinite, $\frac{dv}{d\theta}$ is infinite. That is, the rate of change of velocity is infinite for a finite change in θ. Hence θ has a limiting value which designate i. This is in the ascending branch.

b. Since, so long as θ is positive, $\frac{dv}{d\theta}$ is positive, it follows that v diminishes as θ diminishes. When θ becomes zero, as it does at the summit, we have at that point

$$\left.\frac{dv}{d\theta}\right|_{\theta = 0} = \frac{v\,F(v)}{gC}$$

c. Beyond the summit θ is negative and increases in absolute value, and $\frac{dv}{d\theta}$ will continue positive and v will continue to diminish until $\frac{dv}{d\theta}$ becomes zero and changes sign which occurs when

$$\frac{F(v)}{gC} + \sin\theta = 0$$

and at this point a minimum velocity occurs, its value being such that

$$F(v_m) = -gC\sin\theta_m$$

d. Beyond this point $\frac{dv}{d\theta}$ is negative and v increases as θ diminishes, until θ approaches $-\frac{\pi}{2}$ in value, in which case at the limit

$$\frac{F(v)}{gC} = 1$$

or $$F(v) = g\,C$$

That is the projectile has a limiting velocity v′ such that

$$F(v') = g\,C$$

This is interpreted readily as indicating that the velocity increases until the acceleration due to gravity is balanced by the retardation due to air resistance.

2. Let it be required to trace the curvature of the trajectory in a similar manner.

The radius of curvature, r, of the trajectory is given by the equation

$$\frac{v\,d\theta}{dt} = \frac{v^2}{r} = -g \cos\theta$$

whence

$$r = -\frac{v^2}{g} \sec\theta$$

a. This has infinite values for $v = \infty$ and for $\sec\theta = \infty$; that is, for inclinations i and $-\frac{\pi}{2}$.

Since r is negative the curve is convex upward. Since the radii of curvature at the extremities, (which are infinitely removed from the summit) are infinite it follows that the curve is asymptotic at its extremities.

b. For a minimum radius of curvature

$$\frac{dr}{d\theta} = -\frac{2\,v\,dv}{g\,d\theta} \sec\theta - \frac{v^2}{g} \sec\theta \tan\theta = 0$$

whence

$$\frac{dv}{d\theta} = -\frac{v}{2} \tan\theta$$

but

$$\frac{dv}{d\theta} = v \sec\theta \left(\frac{F(v)}{g\,C} + \sin\theta\right)$$

Hence, at the point of maximum curvature,

$$-\frac{v}{2} \tan\theta_r = v \sec\theta_r \left(\frac{F(v)}{g\,C} + \sin\theta_r\right)$$

or

$$-\sin\theta_r = 2\,\frac{F(v)}{g\,C} + 2\sin\theta_r$$

whence

$$\sin\theta_r = -\frac{2}{3}\,\frac{F(v)}{g\,C}$$

Resumé.

The point of minimum velocity occurs when

$$\frac{F(v_m)}{g\,C} = -\sin\theta_m$$

and that of maximum curvature where

$$\frac{F(v_r)}{g\,C} = -\frac{2}{3} \sin\theta_r$$

Hence they are both in the descending branch of the trajectory, and θ is negative in both cases.

From this we conclude that since at the point of maximum curvature

$$\frac{dv}{d\theta} = -\frac{v}{2}\tan\theta$$

and since $\tan\theta$ is negative and hence $\frac{dv}{d\theta}$ positive, the velocity is still diminishing and hence the point of maximum curvature lies between the summit and point of minimum velocity.

Nomenclature of the Trajectory.

From what has been said, it is clear that the form of the trajectory is as indicated in Figure 2.

At A, $v = \infty,\ \theta = i,\ r = \infty;\ x = -\infty;\ y = -\infty.$

At B, $v = V,\ \theta = \varphi,\ r = -\frac{V^2}{g}\sec\varphi;\ x = 0;\ y = 0.$

At C, $v = v_o.\ \theta = 0,\ r = -\frac{v_o^2}{g};\ x = x_o;\ y = y_o$

At D, $v = v_r,\ \theta = \theta_r,\ r = r_m = -\frac{v_r^2}{g}\sec\theta_r.$

At E, $v = v_m,\ \theta = \theta_m.$

At F, $v = v_\omega,\ \theta = -\omega,\ r = -\frac{v_\omega^2\sec\omega}{g};\ x = X;\ y = 0$

At G, $v = v',\ \theta = -\frac{\pi}{2}, r = -\infty:$

That portion of the trajectory above or near the level BX of the origin, where the muzzle of the gun is assumed to be, is alone considered in practice. The classification of trajectories is based on the values of $\frac{F(v)}{C}$ and θ obtaining in this portion of the trajectory.

Thus Direct Fire embraces a portion of the total trajectory which portion lies in the vicinity of the summit so that both φ and ω are small angles, φ being generally not above 15°.

Curved Fire, which includes also High Angle Fire, embraces all fire with greater elevations. Curved Fire is generally limited in practice to velocities lower than those used in Direct Fire.

As limiting cases two interesting conditions should be first considered. These are rectilinear motion in a resisting medium, taking g = 0; and motion in vacuo, taking C = ∞.

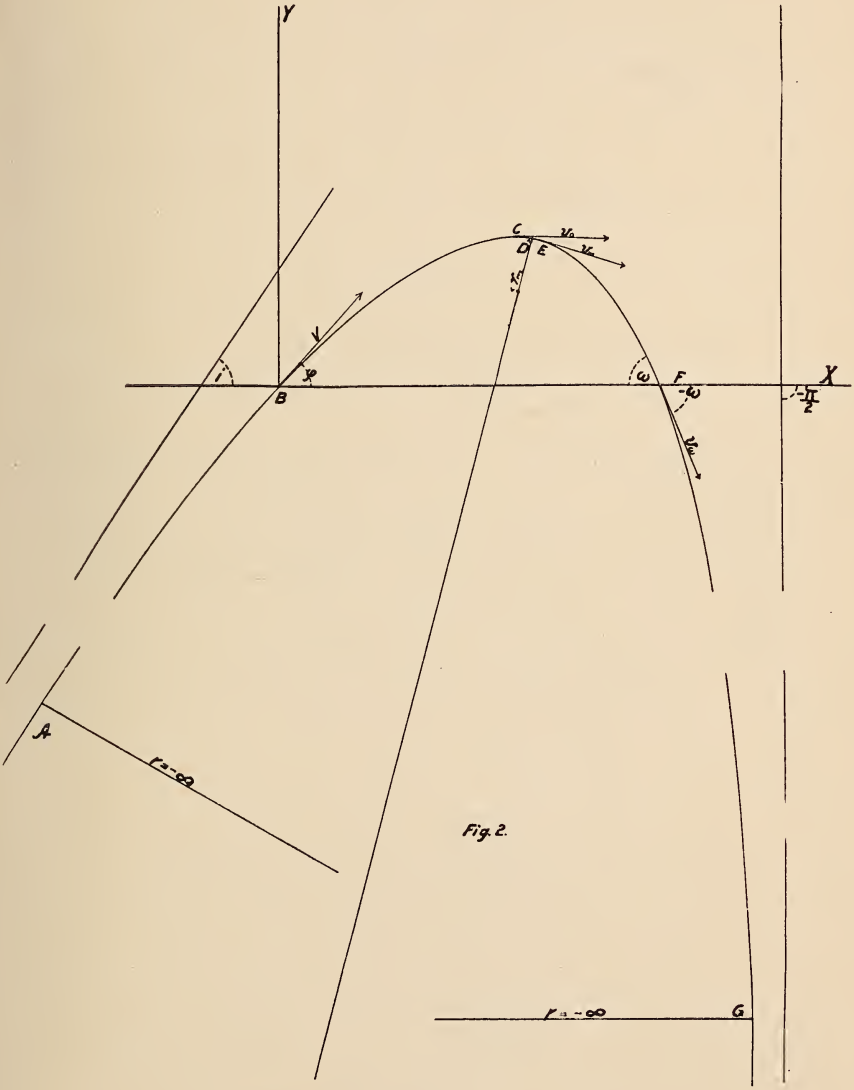
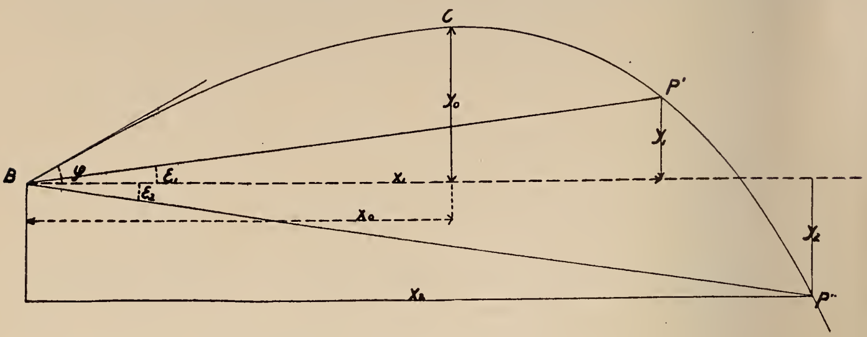


Fig. 2.

As a particular application the case in which F (v) = $A_n V^n$ is considered, and is of great value in connection with curved and high angle fire.

In Figure 3, the trajectory as used in practice is illustrated.



Limiting Cases.

1. $$g = 0$$

The equation of the hodograph becomes

$$\frac{d\theta}{d\,(v\cos\theta)} = 0\,; \quad \text{or} \quad \theta = \varphi\,.$$

Hence, since $ds = vdt$, $dx = \cos\varphi\, ds$; $dy = \tan\varphi\, dx$, it follows that

$$x = s\cos\varphi;\ y = s\sin\varphi$$

We have
$$ds = vdt$$

also
$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = -\frac{F\,(v)}{C}$$

Hence
$$t = C\int_V^v \frac{-\,dv}{F\,(v)} = C\ (T\,(v) - T\,(V))$$

$$s = C\int_V^v \frac{-\,vdv}{F\,(v)} = C\ (S\,(v) - S\,(V)).$$

This is fire at an angle φ with the horizontal. In the case in which $\varphi = 0$, we have horizontal fire and then

$$\frac{t}{C} = T\,(v) - T\,(V)$$

$$\frac{x}{C} = S\,(v) - S\,(V).$$

$$y = 0$$

2. $$C = \infty.$$

The equation of the hodograph becomes

$$\frac{d\,(v\cos\theta)}{d\theta} = 0$$

or
$$v\cos\theta = \text{constant} = V\cos\varphi.$$

Hence $$v = V \cos \varphi \sec \theta$$

Substituting this value of v in expressions for dx, dy, dt, ds, we obtain

$$dt = -\frac{V \cos \varphi}{g} \, d \tan \theta$$

$$dx = -\frac{V^2 \cos^2 \varphi}{g} \, d \tan \theta$$

$$dy = -\frac{V^2 \cos^2 \varphi}{2\,g} \, d\,(\tan^2 \theta),$$

$$ds = -\frac{V^2 \cos^2 \varphi}{g} \sec^3 \theta \, d\theta$$

$$= -\frac{V^2 \cos^2 \varphi}{g} d\ (\theta)_2.$$

in which $(\theta)_2$ or simply (θ) represents

$$\int_0^\theta \sec^3 \theta \, d\theta = \tfrac{1}{2} \left(\tan \theta \sec \theta + \log_e (\tan \theta + \sec \theta)\right)$$

Integrating between θ and φ

$$t = \frac{V \cos \varphi}{g} (\tan \varphi - \tan \theta)$$

$$x = \frac{V^2 \cos^2 \varphi}{g} (\tan \varphi - \tan \theta)$$

$$y = \frac{V^2 \cos^2 \varphi}{2\,g} (\tan^2 \varphi - \tan^2 \theta)$$

$$s = \frac{V^2 \cos^2 \varphi}{g} ((\varphi) - (\theta)).$$

From these by division

$$\frac{t}{V} = \frac{\sin \varphi}{g} \left(1 - \frac{\tan \theta}{\tan \varphi}\right)$$

$$\frac{x}{V^2} = \frac{\sin 2\varphi}{2\,g} \left(1 - \frac{\tan \theta}{\tan \varphi}\right)$$

$$\frac{s}{V^2} = \frac{\sin 2\varphi}{2\,g} \left(\frac{(\varphi)}{\tan \varphi} - \frac{(\theta)}{\tan \theta}\right).$$

$$\frac{y}{V^2} = \frac{\sin^2 \varphi}{2\,g} \left(1 - \frac{\tan^2 \theta}{\tan^2 \varphi}\right)$$

Point of Fall.

Substituting y = 0 in this last equation we have for the point of fall

$$\tan^2 \theta = \tan^2 \varphi$$

or $$\theta = -\varphi = -\omega$$

and this value of θ in the other ratios, gives

$$\frac{T}{V} = \frac{2 \sin \varphi}{g}, \text{ or } T = \frac{2 V \sin \varphi}{g}$$

$$\frac{X}{V^2} = \frac{\sin 2 \varphi}{g}, \text{ or } X = \frac{V^2 \sin 2 \varphi}{g}$$

$$\frac{S}{V^2} = \frac{\sin 2 \varphi}{g} \cdot \frac{(\varphi)}{\tan \varphi}, \text{ or } S = \frac{V^2 \sin 2 \varphi}{g} \cdot \frac{(\varphi)}{\tan \varphi} = X \frac{(\varphi)}{\tan \varphi}$$

Also

$$\frac{V_\omega}{V} = 1$$

$$\omega = \varphi$$

Hence, for the point of fall the following ratios are unity in vacuo

$$\frac{g T}{2 V \sin \varphi} = 1$$

$$\frac{g X}{V^2 \sin 2 \varphi} = 1$$

$$\frac{V_\omega}{V} = 1$$

$$\frac{\tan \omega}{\tan \varphi} = 1$$

From these

$$\frac{X}{V T \cos \varphi} = 1$$

$$\frac{g T^2}{2 X \tan \varphi} = 1$$

Summit.

On substituting $\theta = 0$, in the expressions for t, x, y, s, etc., it is seen that

$$t_0 = \frac{V \sin \varphi}{g} = \frac{1}{2} T$$

$$x_0 = \frac{V^2 \sin 2 \varphi}{2 g} = \frac{1}{2} X$$

$$y_0 = \frac{V^2 \sin^2 \varphi}{2 g} = \frac{1}{8} g T^2 = \frac{1}{4} X \tan \varphi$$

$$s_0 = \frac{V^2 \cos^2 \varphi}{g} \cdot (\varphi) = \frac{S}{2}$$

$$v_0 = V \cos \varphi$$

The following forms of some of the expressions already given are, while true in vacuo, also useful in practice for low velocities and heavy projectiles:

$$y_0 = \frac{1}{8} g T^2 = \frac{1}{8} X (\tan \varphi + \tan \omega),$$

$$v_\omega T \cos \omega \,.\, V T \cos \varphi = X^2$$

$$\frac{x_0}{X} = \frac{\tan \omega}{\tan \varphi + \tan \omega}$$

$$y = \frac{x^2}{X^2} \left(X - x\right) \left(\tan \omega + \left(\frac{X}{x} - 1\right) \tan \varphi\right)$$

Hence for approximate determinations of ω, v_ω, y_0, from observed values of V, φ, T and X, we may proceed to find ω from the relation

$$\tan \omega = \frac{g T^2}{X} - \tan \varphi$$

and then v_ω from

$$v_\omega = \frac{X^2}{T^2} \cdot \frac{\sec \varphi \sec \omega}{V}$$

$$y_0 = \frac{1}{8} g T^2$$

$$x_0 = \frac{X \tan \omega}{\tan \varphi + \tan \omega}$$

Applications.

1. TO FIRE AT A POINT NOT ON THE SAME LEVEL AS THE ORIGIN.

Given V, x, y, to find φ.

$$\tan \varepsilon = \frac{y}{x} = \tan \varphi - \frac{g x}{2 V^2 \cos^2 \varphi} = \tan \varphi \left(1 - \frac{x}{X}\right).$$

Let φ_x be the quadrant angle of departure such that the trajectory due to φ_x and the given V will have its point of fall at x, 0.

Then $$x = \frac{V^2 \sin 2\varphi_x}{g}$$

and $$\frac{x}{X} = \frac{\sin 2 \varphi_x}{\sin 2 \varphi}$$

Hence $$\tan \varepsilon = \tan \varphi \left(1 - \frac{\sin 2 \varphi_x}{\sin 2 \varphi}\right)$$

whence $$\sin 2 \varphi - \sin 2 \varphi_x = 2 \cos^2 \varphi \tan \varepsilon$$

that is $$\sin 2 \varphi_x = \frac{2 \cos \varphi}{\cos \varepsilon} \sin (\varphi - \varepsilon)$$

$$= \frac{2 \cos \varphi \sin (\varphi - \varepsilon)}{\cos \varepsilon}$$

Now, from trigonometry

$$2 \cos \varphi \sin (\varphi - \varepsilon) = \sin (2 \varphi - \varepsilon) - \sin \varepsilon$$

Hence

$$\sin (2 \varphi - \varepsilon) = \cos \varepsilon \,(\tan \varepsilon + \sin 2 \varphi_x).$$

From which $2 \varphi - \varepsilon$ and φ may be found.

2. THE PRINCIPLE OF THE RIGIDITY OF THE TRAJECTORY.

This "principle" assumes that $\varphi - \varepsilon = \varphi_x$. But it has been shown that

$$\sin 2\varphi_x = \frac{2\cos\varphi \sin(\varphi - \varepsilon)}{\cos\varepsilon}.$$

Distinguishing $\varphi - \varepsilon$ as φ' we have, since $\varphi = \varphi' + \varepsilon$,

$$\sin 2\varphi_x = \frac{2\cos(\varphi' + \varepsilon)\sin\varphi'}{\cos\varepsilon} = \frac{2\sin\varphi'\cos\varphi'\cos\varepsilon - 2\sin^2\varphi'\sin\varepsilon}{\cos\varepsilon}$$

$$= \sin 2\varphi'(1 - \tan\varphi'\tan\varepsilon).$$

The assumption that $\varphi' = \varphi - \varepsilon$ is productive of no material error so long as $\tan\varphi'\tan\varepsilon$ is insignificant in comparison with unity. This is generally true in direct fire, but in curved or high angle fire the method of Application 1, should be used as far as practicable.

3. MEAN VALUES.

(a). The Mean Ordinate, h, of the Trajectory.

$$h = \frac{\int_0^s y\,ds}{\int_0^s ds} = \frac{\int_0^s y\sec^3\theta\,d\theta}{\int_0^s \sec^3\theta\,d\theta}$$

but

$$y = \frac{V^2\sin^2\varphi}{2g}\left(1 - \frac{\tan^2\theta}{\tan^2\varphi}\right); \text{ and } y_0 = \frac{V^2\sin^2\varphi}{2g}$$

Hence

$$\frac{y}{y_0} = 1 - \frac{\tan^2\theta}{\tan^2\varphi}$$

Therefore

$$\frac{h}{y_0} = \frac{\int_0^\varphi (1 - \tan^2\theta\cot^2\varphi)\sec^3\theta\,d\theta}{\int_0^\varphi \sec^3\theta\,d\theta}$$

$$= 1 - \frac{1}{4}\left(\frac{\sec^3\varphi}{(\varphi)\tan\varphi} - \cot^2\varphi\right)$$

Taking $y_0 = \frac{1}{4}X\tan\varphi$, it follows that

$$\frac{h}{X} = \frac{1}{4}\left(\tan\varphi - \frac{1}{4}\frac{\sec^3\varphi}{(\varphi)} + \frac{1}{4}\cot\varphi\right)$$

$$= \frac{1}{4}\tan\varphi - \frac{1}{16}\left(\frac{\sec^3\varphi}{(\varphi)} - \cot\varphi\right).$$

Taking $y_0 = \frac{V^2\sin^2\varphi}{2g}$, it follows that

$$\frac{h}{V^2} = \frac{\sin 2\varphi}{4g}\left(\tan\varphi - \frac{1}{4}\frac{\sec^3\varphi}{(\varphi)} + \frac{1}{4}\cot\varphi\right).$$

The mean height is useful in correcting for altitude and is here given in terms of X and φ, and of V and φ. The latter gives too large a value for h but does very well (when X is not known) as a first approximation.

(b). Mean values of $\cos\theta$ and of $\sec\theta$.

$$(\cos\theta)_m = \frac{\int_0^\varphi \cos\theta \sec^3\theta \, d\theta}{(\varphi)} = \frac{\tan\varphi}{(\varphi)} \tag{1}$$

$$(\sec\theta)_m = \frac{\int_0^\varphi \sec\theta \sec^3\theta \, d\theta}{(\varphi)} = \frac{3\tan\varphi + \tan^3\varphi}{3\,(\varphi)} \tag{2}$$

$$\frac{(\sec\theta)_m}{(\cos\theta)_m} = \sec^2\theta'_m = \frac{1}{\cos^2\theta'_m} = \frac{3\tan\varphi+\tan^3\varphi}{3\tan\varphi} = 1+\tfrac{1}{3}\tan^2\varphi \tag{3}$$

or

$$1+\tan^2\theta'_m = 1+\tfrac{1}{3}\tan^2\varphi$$

Hence

$$\tan\theta'_m = \pm\frac{\tan\varphi}{\sqrt{3}}$$

θ'_m is thus an angle for which $(\cos\theta)_m$ and $(\sec\theta)_m$ will consist. It is the best determination for general use. The double sign does not signify as θ'_m is used to obtain only $\cos\theta'_m$ and $\sec\theta'_m$ which are even functions of $\tan\theta$.

4. DROP DUE TO GRAVITY.

(a). Jump.

A projectile emerging from the muzzle of a piece drops, due to the action of gravity, from the straight line $y = x\tan\varphi$, on which it departed, a vertical distance given by the equation of the trajectory in vacuo; thus:

$$y = x\tan\varphi - \tfrac{1}{2}\frac{g\,x^2}{V^2\cos^2\varphi}$$

Hence the drop is $\tfrac{1}{2}\dfrac{g\,x^2}{V^2\cos^2\varphi}$ feet.

The gun being laid by bore-sights on a mark, the height at which the projectile actually strikes must be increased by the drop in order to give a point in prolongation of the line of departure.

This gives the means of determining the amount and direction of the jump.

The drop *in inches* is $\dfrac{6\,g\,x^2}{V^2\cos^2\varphi}$

in which as before x is the distance in feet to the screen on which the jump is measured.

(b) Danger Space.

Using the expression

$$y = \frac{x}{X^2}\,(X - x)\,(x\tan\omega + (X - x)\tan\varphi)$$

and representing X — x, the danger space, by D, there results

$$y = \frac{X-D}{X^2} . D. ((X-D) \tan \omega + D \tan \varphi)$$

or $$y X^2 = D (X-D)^2 \tan \omega + D^2 (X-D) \tan \varphi$$

whence dividing by $X^3 \tan \omega$

$$\frac{y \cot \omega}{X} = \frac{D}{X} - (P+1)\left(\frac{D}{X}\right)^2 + P\left(\frac{D}{X}\right)^3$$

in which $$P = 1 - \frac{\tan \varphi}{\tan \omega}$$

A table of values of $\frac{y \cot \omega}{X}$ may be computed with P and $\frac{D}{X}$ as arguments, and $\frac{D}{X}$ taken out by interpolation using $\frac{y \cot \omega}{X}$ and P as arguments. From this table may be computed $\frac{D}{y \cot \omega}$ for a given P and $\frac{y \cot \omega}{X}$.

Ordinarily, however, the last term is insignificant and the second comparatively so, and the following method may be used:

Take $D = y \cot \omega$ as a first approximation and substitute this value in the last two terms, thus

$$\frac{y \cot \omega}{X} - \frac{D}{X} = (P+1) \frac{y^2 \cot^2 \omega}{X^2} + P \frac{y^3 \cot^3 \omega}{X^3}$$

$$D = y \cot \omega + \left(\frac{P+1}{X}\right) y^2 \cot^2 \omega - \frac{P}{X^2} y^3 \cot^3 \omega = X - x$$

in which $$P = 1 - \frac{\tan \varphi}{\tan \omega}$$

This formala may be used to find x, when y is given, on a given trajectory especially in curved and high angle fire, and is useful for that purpose. Ordinarily, for danger-space problems P may be replaced by 0 giving

$$D = y \cot \omega + \frac{y^2 \cot^2 \omega}{X} = y \cot \omega \left(1 + \frac{y \cot \omega}{X}\right).$$

5. MAXIMUM AND MINIMUM VALUES.

(a). Range, as a function of Elevation.

$$X = \frac{V^2 \sin 2 \varphi}{g}$$

$$\frac{d X}{d \varphi} = \frac{V^2}{g} . 2 \cos 2 \varphi = 0 \text{ for a max.}$$

Hence $$\cos 2 \varphi = 0; 2 \varphi = \frac{\pi}{2}; \varphi = \frac{\pi}{4} = 45°$$

Hence, in vacuo, 45° is the angle of greatest range. This is also nearly true in curved fire, the value of φ being somewhat under 45°.

(b). Time of Flight, as a function of Elevation.

$$T = \frac{2\,V \sin \varphi}{g}$$

$$\frac{dT}{d\varphi} = \frac{2V}{g} \cos \varphi = 0$$

Hence $$\cos \varphi = 0 \text{ and } \varphi = \frac{\pi}{2} = 90°.$$

(c). Minimum velocity in the trajectory due to V, φ.

$$v = V \cos \varphi \sec \theta$$

$$\frac{dv}{d\theta} = V \cos \varphi \sec \theta \tan \theta = 0 \text{ for min.}$$

Hence $\sec \theta \tan \theta = 0$ or $\theta = 0$.

Hence minimum velocity is at the summit. This is nearly true in curved and high angle fire with low velocities, but is not even approximately true in direct fire.

(d). Maximum Curvature, as a function of θ.

We have $$\frac{v^2}{r} = -g \cos \theta \text{ or } r = -\frac{v^2}{g \cos \theta} = -\frac{V^2 \cos^2 \varphi \sec^2 \theta}{g \cos \theta}$$

Hence $$r = -\frac{V^2 \cos^2 \varphi}{g} . \sec^3 \theta$$

and

$$\frac{dr}{d\theta} = -\frac{V^2 \cos^2 \varphi}{g} . 3 \sec^3 \theta \tan \theta = 0$$ for min. r or max. curvature, which gives $\theta = 0$.

or the maximum curvature occurs at the summit.

This is not absolutely true in air, but is nearly so in curved and high angle fire.

These maxima and minima serve as points of departure in seeking those in air, rendering fewer approximations necessary.

DIRECT FIRE.

This class of fire pertains to small angles of departure, generally not above 15°, though it may be used for greater elevations with the introduction of a suitable factor in the ballistic coefficient.

The method here used is that of Siacci, in which is used the preudo-velocity which is designated u, and which fulfils the condition

$$u \cos \varphi = v \cos \theta$$

or $$u = v \cos \theta \sec \varphi$$

The integration of the hodograph is accomplished by the use of an integrating factor, β, such that

$$F(v) = \beta . F(u) \cos^2\varphi \sec\theta,$$

with the practical certainty that, for direct fire, the value of β will never differ greatly from unity. In any case, however, we may write the equation just assumed.

$$\frac{F(v)}{C} = \frac{\beta}{C} . F(u) \cos^2\varphi \sec\theta = \frac{F(u)}{C_s} \cos^2\varphi \sec\theta$$

in which $C_s = \frac{C}{\beta}$.

We will, however, use C instead of C_s with the understanding that where Siacci's method is used, C is to be divided by β.

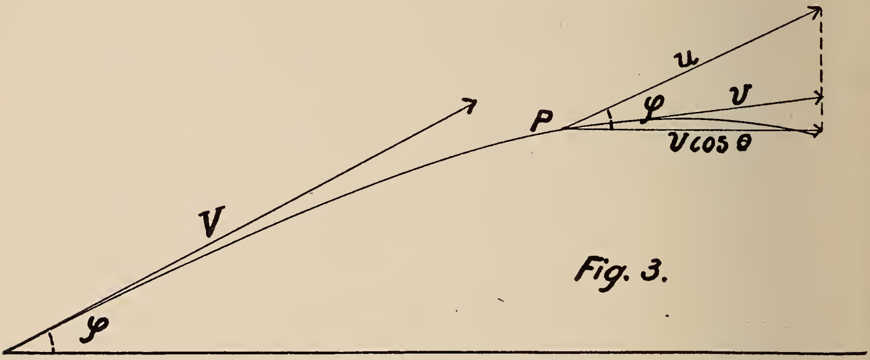


Fig. 3.

Figure 3 shows the character of u. It is thus seen that u is a velocity parallel to the muzzle velocity and having the same horizontal component as the actual velocity v.

The equation of the hodograph becomes, under these conditions

$$\frac{d(u\cos\varphi)}{d\theta} = \frac{u\cos\varphi\sec\theta . F(u) . \cos^2\varphi\sec\theta}{g C}$$

or

$$\frac{du}{u F(u)} = \frac{\cos^2\varphi\sec^2\theta\, d\theta}{g C} = \frac{\cos^2\varphi}{g C} . d\tan\theta$$

or, integrating between θ and φ

$$\tan\theta - \tan\varphi = \frac{-C}{2\cos^2\varphi}\int_V^u \frac{-2 g\, du}{u F(u)} = -\frac{C}{2\cos^2\varphi}\Big(I(u) - I(V)\Big)$$

The expressions for dt, dx, dy, may be written

$$dt = -\frac{u \cos \varphi}{g} . \; d \tan \theta = - C \sec \varphi \frac{du}{F(u)}$$

$$dx = u \cos \varphi \; dt = - \frac{Cu \; du}{F(u)}$$

$$dy = u \cos \varphi \tan \theta \; dt = - \frac{Cu \; du}{F(u)} \left(\tan \varphi - \frac{C}{2 \cos^2 \varphi} (I(u) - I(V)) \right)$$

and from these

$$t = C \sec \varphi \int_V^u - \frac{du}{F(u)} = C \sec \varphi \Big(T(u) - T(V) \Big)$$

$$x = C \int_V^u - \frac{u \; du}{F(u)} = C \Big(S(u) - S(V) \Big) ..$$

$$y = x \tan \varphi - \frac{C^2}{2 \cos^2 \varphi} \int_V^u (I(u) - I(V)) \, d \, S(u).$$

$$= x \tan \varphi - \frac{C^2}{2 \cos^2 \varphi} \Big(A(u) - A(V) - I(V)(S(u) - S(V)) \Big)$$

in which

$$A(u) = \int I(u) \; d \, S(u)$$

Noting that $S(u) - S(V) = \frac{x}{C}$ and dividing both members of the equation for y by x,

$$\tan \varepsilon = \frac{y}{x} = \tan \varphi - \frac{C}{2 \cos^2 \varphi} \left(\frac{A(u) - A(V)}{S(u) - S(V)} - I(V) \right)$$

The angle ε as thus defined is the position angle. In Part I, its absolute value alone was considered, but here it is of the same sign as y.

The S, A, I and T functions of u constitute the principal ballistic functions and are in general use in all the principal countries in the world. From a table of these functions certain combinations were derived by Major Braccialini and tabulated. These functions materially abridge the labor of computation, and are tabulated with V and $\frac{X}{C}$ as arguments. Of course the values of all these functions will depend on the law of resistance adopted. That in use in this country and forming the basis of Ingalls' Tables I and II is due to General Mayevski, is based on the Krupp firings at Meppen and was extended by Colonel Zabudski to include velocities up to about 3600 f.s.*

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* See Appendix III.

Braccialini's secondary functions are as follows:

$$t' = \frac{t}{C} \cos \varphi = T\ (u) - T\ (v).$$

$$a' = \frac{2 \cos^2 \varphi\ (\tan \varphi - \tan \theta)}{C} = I(u) - I\ (V)$$

$$a = \frac{2 \cos^2 \varphi\ (\tan \varphi - \tan \varepsilon)}{C} = \frac{A(u) - A(V)}{S(u) - S(V)} - I\ (V)$$

$$b = \frac{2 \cos \varphi\ \ (\tan \varepsilon - \tan \theta)}{C} = a' - a$$

$$z = \frac{x}{C} = S\ (u) - S\ (V)$$

$$c' = \frac{2 \cos^2 \varphi\ (\tan \varphi - \tan \varepsilon)}{x} = \frac{a}{z}$$

$$b' = \frac{\tan \varphi - \tan \varepsilon}{\tan \varepsilon - \tan \theta} = \frac{b}{a}$$

The type-symbols of these functions are not in all cases the same as those used by Braccialini, but are those used by Colonel Ingalls and are here given so that there will be no confusion in using Tables I and II, Artillery Circular M.

Using the secondary functions, working equations take the form

$$\tan \theta = \tan \varphi - \frac{a'\ C}{2 \cos^2 \varphi} = \tan \varphi \left(1 - \frac{a'\ C}{\sin 2\ \varphi}\right)$$

$$t = Ct' \sec \varphi$$

$$\tan \varepsilon = \frac{y}{x} = \tan \varphi - \frac{a\ C}{2 \cos^2 \varphi} = \tan \varphi \left(1 - \frac{a\ C}{\sin 2\ \varphi}\right)$$

$$\frac{\tan \varphi - \tan \varepsilon}{\tan \varepsilon - \tan \theta} = b'$$

$$\frac{\sin 2\ \varphi}{x} \left(1 - \frac{\tan \varepsilon}{\tan \varphi}\right) = c'$$

$$x = C\ z$$

The function a′ has another use, shown by Colonel Ingalls. It is as follows:

For the summit, $\theta = o$; hence, at the summit,

$$a' = \frac{\sin 2\ \varphi}{C}$$

It is also true that, at the point of fall, where $y = o$,

$$a = \frac{\sin 2\ \varphi}{C}$$

Affecting the letters referring to the summit with a zero subscript, and using the corresponding capital letters for the point of fall, these two conditions are expressed as follows:

$$a_0' = \frac{\sin 2\ \varphi}{C} = A$$

The condition $a_0' = A$ at once determines a_0' when A is known, and Col. Ingalls used this relation in determining a function a_0'' for the determination of the maximum ordinate. Thus

$$\frac{y_0}{x_0} = \tan\varphi\left(1 - \frac{a_0}{A}\right) = \tan\varphi\,\frac{a_0' - a_0}{a_0'} = \frac{b_0}{a_0'}\tan\varphi$$

whence

$$\frac{y_0}{C\tan\varphi} = \frac{x_0}{C}\cdot\frac{b_0}{a_0'} = \frac{b_0\,z_0}{a_0'} = a_0''$$

So that $y_0 = a_0''\,C\tan\varphi$.

Thus, in using the tables, a_0' is at once known when A is known and all formulas relating to the summit may be used on taking out the necessary quantities by interpolation using a_0' as argument.

The calculation of the functions a, a′, a″, b, b′, c′, t′ is undertaken with assumed values of z at regular intervals and with assumed values of V at intervals allowing satisfactory interpolation.

Thus with assumed values of V and z, u is at once found by means of Table I from the relation

$$S(u) = z + S(V).$$

The S, A, I and T functions of u and V are to be found in Table I and the excess of the functions of u over the corresponding functions of V are found. Representing these excesses by $\triangle_S$, $\triangle_A$, $\triangle_I$, $\triangle_T$, so that

$$\begin{aligned}\triangle_S &= S(u) - S(V)\\ \triangle_A &= A(u) - A(V)\\ \triangle_I &= I(u) - I(V)\\ \triangle_T &= T(u) - T(V)\end{aligned}$$

it follows that,

$$\begin{aligned}a &= \frac{1}{z}\triangle_A - I(V)\\ a' &= \triangle_I\\ t' &= \triangle_T\end{aligned}$$

The remaining functions are readily found when a′, a and z are known, thus

$$\begin{aligned}b &= a' - a\\ b' &= \frac{b}{a}\\ c' &= \frac{a}{z}\\ a'' &= \frac{b\,z}{a'}\end{aligned}$$

Point of Fall.

The equations for the point of fall, where $y = o$, become

$$X = C\,Z$$
$$T = C\,T' \sec\varphi$$
$$\sin 2\varphi = A\,C$$
$$\frac{\tan\omega}{\tan\varphi} = B'$$
$$V_\omega = u_\omega \cos\varphi \sec\omega$$

Summit.

Using $a_0' = A$, take out z_0, a_0'' and any other desired functions for the summit, then

$$x_0 = C z_0$$
$$y_0 = a_0''\,C \tan\varphi$$
$$t_0 = C t_0' \sec\varphi$$
etc.

Any Point on the Trajectory.

$$\tan\theta = \frac{\tan\varphi}{A}\left(A - a'\right)$$
$$\tan\varepsilon = \frac{y}{x} = \frac{\tan\varphi}{A}\left(A - a\right)$$

The other elements may be found in the same manner as for the point of fall using the proper value of z as argument. It is a property of pseudo-velocity that for a given V and Z it is independent of the angle of departure φ or of the height y. This is true because

$$S\,(u) = z + S\,(V)$$

showing that u is dependent on z and V alone, and since the secondary functions are dependent only on Z and V, they are all independent of φ or y.

Represent by φ_x the value of φ which would for a given V and z cause the point of fall to occur at x, o, the foot of the ordinate y, whose abscissa is x. In that case ε becomes zero and hence

$$\sin 2\varphi_x = a\,C$$

It is also true that

$$a\,C = 2\cos^2\varphi\,(\tan\varphi - \tan\varepsilon) = \sin 2\varphi\left(1 - \frac{\tan\varepsilon}{\tan\varphi}\right)$$

Hence

$$\sin 2\varphi_x = \sin 2\varphi - 2\cos^2\varphi \tan\varepsilon$$

This is exactly the equation already found for conditions in vacuo, and hence

$$\sin 2\varphi_x = \frac{2\cos\varphi\sin(\varphi - \varepsilon)}{\cos\varepsilon}$$

and

$$\sin (2\varphi - \varepsilon) = \cos \varepsilon\, (\tan \varepsilon + \sin 2\varphi_x) = \cos \varepsilon\, (a\, C + \tan \varepsilon)$$

which may be solved for φ, when V, x, y and C are given.

Since the equations are identical in form with those for conditions in vacuo, the error in elevation due to the assumption of the principle of rigidity of the trajectory may be found from the relation

$$\sin 2\varphi_x = \sin 2\varphi' \,(1 - \tan \varphi' \tan \varepsilon).$$

Forms Analogous to those in Vacuo.

The equations already found may be placed in forms analogous to those in vacuo; thus

$$y = x \tan \varphi - \frac{g\, x^2}{2 V^2 \cos^2 \varphi} q_y = x \tan \varphi \left(1 - \frac{g\, x\, q_y}{V^2 \sin 2\varphi}\right)$$

$$\tan \theta \doteq \tan \varphi - \frac{g\, x}{V^2 \cos^2 \varphi} \,.\, q_\theta = \tan \varphi \left(1 - \frac{2\, g\, x\, q_\theta}{V^2 \sin 2\varphi}\right)$$

$$t = \frac{x}{V \cos \varphi} \,.\, q_t$$

$$v \cos \theta = V \cos \varphi .\, q_v .$$

in which

$$q_y = \frac{V^2 \sin 2\varphi}{g\, x} \,.\, \frac{a\, C}{\sin 2\varphi} = \frac{a\, V^2}{g\, z} = \frac{V^2\, c'}{g}$$

$$q_\theta = \frac{V^2 \sin 2\varphi}{2\, g\, x} \,.\, \frac{a\, C}{\sin 2\varphi} = \frac{a'\, V^2}{2\, g\, Z}$$

$$q_t = \frac{V \cos \varphi}{x} \,.\, c t' \sec \varphi = \frac{V t'}{Z}$$

$$q_v = \frac{v \cos \theta \sec \varphi}{V} = \frac{u}{V}$$

The form q_y is exceedingly useful in connection with experimental firings to determine c. It varies less rapidly with V than does c′ and interpolation is facilitated by its use.

All these functions are especially useful in the case in which $F(v) = A_n v^n$, as will now be shown.

CASE OF $F(v) = A_n v^n$.

Representing by m the product $A_n V^{n-2} z$ and noting that

$$S(u) - S(V) = \frac{1}{(n-1) A_n V^{n-2}} \left(\left(\frac{V}{u}\right)^{n-2} - 1\right) = z$$

$$T(u) - T(V) = \frac{1}{(n-1) A_n V^{n-1}} \left(\left(\frac{V}{u}\right)^{n-1} - 1\right)$$

$$I(u) - I(V) = \frac{2\, g}{n\, A_n V^n} \left(\left(\frac{u}{V}\right)^{n} - 1\right)$$

$$A(u) - A(V) = \frac{g}{n\,(n-1)\, A_n^2\, V^{2n-2}} \left(\left(\frac{V}{u}\right)^{2n-2} - 1\right)$$

it follows that

$$m = \frac{1}{n-2}\left(\left(\frac{V}{u}\right)^{n-2} - 1\right)$$

$$q_y = \frac{1}{n(n-1)m^2}\left((1+(n-2)m)^{\frac{2n-2}{n-2}} - 1 - 2(n-1)m\right)$$

$$q_\theta = \frac{1}{n\,m}\cdot\left((1+(n-2)m)^{\frac{n}{n-2}} - 1\right)$$

$$q_t = \frac{1}{(n-1)m}\left((1+(n-2)m)^{\frac{n-1}{n-2}} - 1\right)$$

$$q_v = (1+(n-2)m)^{\frac{-1}{n-2}}$$

The principal entire values of n are 2, 3, 4; and for these the following forms are taken by the functions.

n	2	3	4
m	$A_2\,z$	$A_3\,Vz$	$A_4\,V^2z$
q_y	$e^{2m} - 1 - 2m$	$1+\frac{2}{3}m+\frac{1}{6}m^2$	$1+\frac{2}{3}m$
q_θ	$\frac{e^{2m}-1}{2m}$	$1+m+\frac{1}{3}m^2$	$1+m$
q_t	$\frac{e^{m}-1}{m}$	$1+\frac{1}{2}m$	$\frac{(1+2m)^{\frac{3}{2}}-1}{3m}$
q_v	e^{-m}	$\frac{1}{1+m}$	$\frac{1}{(1+2m)^{\frac{1}{2}}}$

From the expression for q_y it is seen that the trajectory is a curve whose degree is $\frac{2n-2}{n-2}$, and hence the degree of the trajectory for the following values of n is as indicated.

n	degree	F(v)	Condition
$-\infty$	2	0	Vacuum
0	*	A_0	Const. resistance
1	*	$A_1\,v$	Linear
2	∞	$A_2\,v^2$	Quadratic
3	4	$A_3\,v^3$	Cubic
4	3	$A_4\,v^4$	Biquadratic
$\frac{8}{3}$	5		
2	6		
$\frac{12}{5}$	7		
$\frac{7}{3}$	8		
$\frac{2-2m}{2-m}$	m	$A_n\,v^n$	

It will be noted that, for n = 2 and below, the trajectory is transcendental. For $n = 2$ the familiar limit $(1+p)^{1/p}$ on p approaching zero, occurs, and is in the form

$$(1+(n-2)m)^{\frac{1}{n-2}}\Big]_{n-2=0} = \text{the Napierian base } e.$$

In the case $n = 1$ and $n = 0$ illusory forms occur; but, on evaluation, show logarithmic forms. Thus:

n	0	1
m	A_0	$A_1\ v$
q_y	$-\frac{2}{m^2}((1-2m)\log_e(1-2m)-2m)$	$-\frac{2}{m}\left(1+\frac{\log_e(1-m)}{m}\right)$
q_θ	$-\frac{1}{2}\log_e(1-2m)$	$\frac{1}{1-m}$
q_t	$-\frac{1}{m}\left((1-2m)^{\frac{1}{2}}-1\right)$	$-m\log_e(1-m)$
q_v	$(1-2m)^{\frac{1}{2}}$	$1-m$

Expansion in series, for the general exponent n, gives the following:

$$q_y = 1+\frac{4}{\underline{|3}}\,m+\frac{4}{\underline{|4}}\,(4-n)\,m^2+\frac{4}{\underline{|5}}\,(4-n)\,(6-2n)\,m^3+\text{etc.}$$

$$q_\theta = 1+\frac{2}{\underline{|2}}\,m+\frac{2}{\underline{|3}}\,(4-n)\,m^2+\frac{2}{\underline{|4}}\,(4-n)\,(6-2n)\,m^3+\text{etc.}$$

$$q_t = 1+\frac{1}{\underline{|2}}\,m+\frac{1}{\underline{|3}}\,(3-n)\,m^2+\frac{1}{\underline{|4}}\,(3-n)\,(5-2n)\,m^3+\text{etc.}$$

$$q_v = 1-\frac{1}{\underline{|1}}\,m-\frac{1}{\underline{|2}}\,(1-n)\,m^2-\frac{1}{\underline{|3}}\,(1-n)\,(3-2n)\,m^3+\text{etc.}$$

These series will have a finite number of terms for values of n as follows, representing the number of terms in the series by r:

For q_y and q_θ, $n=\frac{2r}{r-1}$; for q_t, $n=\frac{2r-1}{r-1}$; for q_v, $n=\frac{2r-3}{r-1}$.

THE CONSTRUCTION OF TABLES FOR THE CASE $F(v) = A_n V^n$.

With m and n as arguments the values of the q-factors or their logarithms may be readily calculated, and tabulated, together with such additional factors as may be derivable from them, using the formulas already deduced.

There is one case remarkable on account of the fact that V is not involved in m. This is the case in which $n = 2$, and, as a consequence,

$$m = A_2\, z$$

This is useful in curved fire as it is nearly true, for the low velocities used, that the quadratic law obtains. Knowing A_2 we may use $\frac{m}{A_2}$ or z as the sole argument of the table.

Quadratic Law of Resistance.

Placing

$$q_1 = g\,q_y;\ q_2 = \frac{2\,q_\theta}{q_y} - 1;\ q_3 = \frac{q_1}{2\,q_t};\ q_4 = \frac{1}{q_v}.$$

$$q_5 = z\,q_1;\ q_6 = \frac{q_5\,q_2}{q_4};\ q_7 = \frac{1}{2\,A_2}\log_e q_\theta$$

We then have

$$y = x \tan \varphi - \frac{x^2\,q_1}{2\,V^2 \cos^2\varphi}$$

$$\tan \theta = \tan \varphi - \frac{x\,q_1}{V^2 \cos^2 \varphi} \cdot \frac{q_2 + 1}{2}$$

$$t = \frac{x}{V \cos \varphi} \cdot \frac{q_1}{2\,q_3}$$

$$v = \frac{V \cos \varphi \sec \theta}{q_4}$$

$$\frac{2\,V^2 \cos \varphi \sin\,(\varphi - \varepsilon)}{C \cos \varepsilon} = q_5$$

$$\frac{2\,V^2 \cos \theta \sin\,(\varepsilon - \theta)}{C \cos \varepsilon} = q_6$$

$$z_0 = Q_7$$

$$\frac{y_0}{X \tan \varphi} = \frac{z_0}{Z}\left(1 - \frac{z_0}{Z} \cdot \frac{q_1\,(z_0)}{Q_1}\right) = Q_8$$

in which $q_1\,(z_0)$ is the value of q_1 for $z = z_0$ and Q_1 refers to the point of fall.

POINT OF FALL.

At the point of fall ε becomes zero and

$$\frac{V^2 \sin 2\,\varphi}{X} = Q_1$$

$$\frac{\tan \omega}{\tan \varphi} = Q_2$$

$$\frac{V \sin \varphi}{T} = Q_3$$

$$\frac{V}{u_\omega} = Q_4$$

$$\frac{V^2 \sin 2\,\varphi}{C} = Q_5$$

$$\frac{v^2_\omega \sin 2\,\omega}{C} = Q_6$$

$$z_0 = Q_7$$

$$\frac{y_0}{X \tan \varphi} = Q_8$$

These together with the equations for the general point, x, y, of the trajectory will give a complete solution for this law.

ANY POINT X, Y, OF THE TRAJECTORY.

Since, as already shown for Siacci's method, the factors depend on $\frac{x}{C}$ and V alone, it follows that the solution for any point may be readily accomplished by the use of the equations already given and by the equations

$$\sin (2\varphi - \varepsilon) = \cos \varepsilon \, (\tan \varepsilon + \sin 2\varphi_x)$$

$$\sin 2\varphi_x = \frac{2 \cos \varphi \sin (\varphi - \varepsilon)}{\cos \varepsilon} = \frac{x}{V^2} \cdot q_1$$

From these and the preceding are found

$$\frac{y}{x} = \tan \varphi \left(1 - \frac{\sin 2\varphi_x}{\sin 2\varphi}\right)$$

$$\tan \theta = \tan \varphi \left(1 - (q_2+1) \frac{\sin 2\varphi_x}{\sin 2\varphi}\right).$$

$$t \cos \varphi = \frac{V}{2} \cdot \frac{\sin 2\varphi_x}{q_3}$$

$$v = \frac{V \cos \varphi \sec \theta}{q_4}$$

The Integrating Factor.

It must be remembered that the relation

$$F(v) = \beta F(u) \cos^2 \varphi \sec \theta$$

becomes for $F(v) = A_n v^n$,

$$A_n v^n = \beta A_n u^n \cos^2 \varphi \sec \theta$$

so that

$$\beta = \left(\frac{v}{u}\right)^n \cos \theta \sec^2 \varphi = \cos^{n-2} \varphi \sec^{n-1} \theta$$

For curved fire it is necessary to ascertain and use the value of β. Using $n = 2$ we find

$$\beta = \sec \theta$$

Its mean value is readily found over the range (θ). Thus

$$\frac{1}{\beta} = \frac{\int_{-\omega}^{\varphi} \cos \theta \sec^3 \theta \, d\theta}{(\varphi) + (\omega)} = \frac{\tan \varphi + \tan \omega}{(\varphi) + (\omega)}$$

Finding also the mean value of β in order to determine a mean mean value for β, thus

$$\frac{\int_{-\omega}^{\varphi} \sec^4 \theta \, d\theta}{(\varphi) + (\omega)} = \frac{3 (\tan \varphi + \tan \omega) + (\tan^3 \varphi + \tan^3 \omega)}{3 ((\varphi) + (\omega))}$$

we have

$$\beta^2 = \frac{3\,(\tan\varphi + \tan\omega) + \tan^3\varphi + \tan^3\omega}{3\,(\tan\varphi + \tan\omega)}$$
$$= 1 + \tfrac{1}{3}\,(\tan^2\varphi - \tan\varphi\,\tan\omega + \tan^2\omega)$$

Hence

$$\beta = \sqrt{1 + \tfrac{1}{3}\tan^2\varphi\left(1 - \frac{\tan\omega}{\tan\varphi} + \frac{\tan^2\omega}{\tan^2\varphi}\right)}$$

A fair mean value of β may be gotten by assuming for curved fire the mean ratio $\dfrac{\tan\omega}{\tan\varphi} = 6/5$, which gives practically

$$\beta = \sqrt{1 + \tfrac{2}{5}\tan^2\varphi}$$

This applies with great accuracy to the tables of Curved Fire, giving results almost identical with those obtained by quadratures. It applies to all elevations up to about 60° and performs its function of producing the same range with the same V, φ, and projectile when computation is made by the two methods. It has no other function.

Differential Formulas: General.

It is often desirable to find what effect a change in V, φ, or C will produce on the range to the point of fall.

The relation

$$A = \frac{A\,(u) - A\,(V)}{Z} - I\,(V)$$

or

$$(A + I\,(V))\,Z = A\,(u) - A\,(V)$$

gives on differentiating with respect to A, Z, and S(V),

$$I\,(u)\,dZ + A'\,dS\,(V) = Z\,dA + A\,dZ + I\,(V)\,dZ + Z\,dI\,(V)$$

or

$$(A' - A)\,dZ + A'\,dS\,(V) = Z\,(dI\,(V) + dA)$$

But

$$dI\,(V) = \frac{2g}{V^2}\,dS\,(V) = -\frac{2g\,dV}{V\,F\,(V)}$$

$$dS\,(V) = -\frac{V\,dV}{F\,(V)}$$

$$A' - A = B$$

Hence

$$B\,dZ - \frac{A'\,V^2}{F\,(V)} \cdot \frac{dV}{V} = Z\left(dA - \frac{2g}{F\,(V)} \cdot \frac{dV}{V}\right).$$

whence

$$\frac{dZ}{Z} = \frac{A'\,V^2}{Z\,B\,F\,(V)} \cdot \frac{dV}{V} + \frac{dA}{B} - \frac{2g}{B\,F\,(V)} \cdot \frac{dV}{V}$$
$$= \frac{A'\,V^2 - 2gZ}{Z\,B\,F\,(V)} \cdot \frac{dV}{V} + \frac{dA}{B}$$

Now $\quad Z = \dfrac{X}{C}$ and hence $\dfrac{dZ}{Z} = \dfrac{dX}{X} - \dfrac{dC}{C}$

also

$$\frac{dA}{B} = \frac{A}{B} \cdot \frac{dA}{A} = \frac{A}{B}\left(\frac{d\sin 2\varphi}{\sin 2\varphi} - \frac{dC}{C}\right)$$

$$\text{since } A = \frac{\sin 2\varphi}{C}$$

Hence

$$\frac{dX}{X} - \frac{dC}{C} = \frac{A' V^2 - 2gZ}{Z B F(V)} \cdot \frac{dV}{V} + \frac{A}{B}\left(\frac{d\sin 2\varphi}{\sin 2\varphi} - \frac{dC}{C}\right)$$

or

$$\frac{dX}{X} = \left(1 - \frac{A}{B}\right)\frac{dC}{C} + \frac{A}{B} \cdot \frac{d\sin 2\varphi}{\sin 2\varphi} + \frac{A' V^2 - 2gZ}{Z B F(V)} \cdot \frac{dV}{V}$$

and since $\frac{A}{B} = \frac{\tan\varphi}{\tan\omega}$, and $\frac{A'}{B} = 1 + \frac{\tan\varphi}{\tan\omega}$ and

$$B = A B' = \frac{\sin 2\varphi}{C} \cdot \frac{\tan\omega}{\tan\varphi}\ ;\ \text{also } B' = \frac{\tan\omega}{\tan\varphi}$$

the equation becomes

$$\frac{dX}{X} = \left(1 - \frac{\tan\varphi}{\tan\omega}\right)\frac{dC}{C}$$
$$+ \frac{\tan\varphi}{\tan\omega} \cdot \frac{d\sin 2\varphi}{\sin 2\varphi}$$
$$+ \frac{C}{X}\,\frac{V^2}{F(V)}\left(1 + \frac{\tan\omega}{\tan\varphi} - \frac{2gX}{V^2 \sin 2\varphi}\right)\frac{\tan\varphi}{\tan\omega} \cdot \frac{dV}{V}$$

which may be used directly with a range-table if tabulated values of $\frac{V^2}{F(V)}$ are at hand, or if the value has been found for the particular range-table.

The method of finding range changes by increments is given in Part I.

Special Differential Formulas for Range Changes.

Taking the development

$$Q_y = \frac{V^2 \sin 2\varphi}{gX} = 1 + \tfrac{2}{3}\, m + \text{etc}$$

restricting it to two terms and noting that $m = A_n V^{n-2}\frac{X}{C}$ it becomes

$$\frac{V^2 \sin 2\varphi}{gX} = 1 + \tfrac{2}{3} A_n V^{n-2}\frac{X}{C}$$

Differentiating, it is found that

$$\frac{V^2 \sin 2\varphi}{gX}\left(\frac{2dV}{V} + \frac{d\sin 2\varphi}{\sin 2\varphi} - \frac{dX}{X}\right) = \tfrac{2}{3} A_n V^{n-2}\frac{X}{C}\left((n-2)\frac{dV}{V} + \frac{dX}{X} - \frac{dC}{C}\right) = \left(\frac{V^2 \sin 2\varphi}{gX} - 1\right)\left((n-2)\frac{dV}{V} + \frac{dX}{X} - \frac{dC}{C}\right)$$

Representing $\frac{V^2 \sin 2\varphi}{g X} - 1$ by N, and collecting terms

$$\left(2+(4-n)N\right)\frac{dV}{V} + (N+1)\frac{d \sin 2\varphi}{\sin 2\varphi} + N\frac{dC^*}{C} = (2N+1)\frac{dX}{X}$$

or

$$\frac{dX}{X} = \left(2 - \frac{nN}{2N+1}\right)\frac{dV}{V} + \frac{N+1}{2N+1} \cdot \frac{d \sin 2\varphi}{\sin 2\varphi} + \frac{N}{2N+1}\frac{dC}{C}$$

n† is approximately 2 for long ranges with muzzle velocities above 1800. For field guns with velocities about 1700, n = 3½. In general, for a range table, n may be taken as k+l R, R being the range in yards; and k and l constants giving proper X changes due to V changes at long and short ranges.

In these formulas it is necessary to know only V, φ, and X, in order to apply them, provided n is also known approximately.

For curved fire n = 2 giving

$$\frac{dX}{X} = \frac{2N+2}{2N+1} \cdot \frac{dV}{V} + \frac{N+1}{2N+1} \cdot \frac{d \sin 2\varphi}{\sin 2\varphi} + \frac{N}{2N+1} \cdot \frac{dC}{C}$$

For field Artillery, V between 1000 and 1700, n = 3½, hence

$$\frac{dX}{X} = \frac{N+4}{4N+2} \cdot \frac{dV}{V} + \frac{N+1}{2N+1} \cdot \frac{d \sin 2\varphi}{\sin 2\varphi} + \frac{N}{2N+1} \cdot \frac{dC}{C}$$

The differential formulas apply to small changes in range due to changes in the different elements. They are useful for quick computations using only a range table and a table of logarithms; they are also of value in showing what will result if a change is made from one gun to another, as, for instance, when it is desired to find what change is to be made in muzzle velocity when the gun and projectile are changed, in order to acquire the same range with the same angle of departure. In that case (using for illustration our approximate formulas with properly determined n), $\frac{dX}{X}$ and $\frac{d \sin 2\varphi}{\sin 2\varphi}$ become zero and hence

$$\left(\frac{2+(4-n)N}{2N+1}\right)\frac{dV}{V} + \frac{N}{2N+1}\frac{dC}{C} = 0$$

whence

$$\frac{dV}{V} = -\frac{N}{2+(4-n)N} \cdot \frac{dC}{C}$$

For the same range and a different V and C to find the change in φ:

---

* See Appendix II.

† See Appendix III.

$$\frac{d \sin 2\varphi}{\sin 2\varphi} = -\frac{2N+1}{N+1} \times \frac{2+(4-n)N}{2N+1} \cdot \frac{dV}{V} - \frac{2N+1}{N+1} \cdot \frac{N}{2N+1} \frac{dC}{C}$$

whence

$$\frac{d \sin 2\varphi}{\sin 2\varphi} = -\frac{2+(4-n)N}{N+1} \cdot \frac{dV}{V} - \frac{N}{N+1} \cdot \frac{dC}{C}$$

that is

$$d \log \sin 2\varphi = -\frac{2+(4-n)N}{N+1} \cdot d \log V - \frac{N}{N+1} \cdot d \log C$$

These are illustrations only; other applications will suggest themselves.

THE SOLUTION BY QUADRATURES IN THE CASE $F(v) = A_n v^n$

The equation of the hodograph becomes

$$\frac{d\,(v \cos \theta)}{d\theta} = \frac{v\,F(v)}{g\,C} = \frac{A_n\, v^{n+1}}{g\,C}$$

Placing $v \cos \theta \sec \varphi = u$, we have

$$\cos \varphi \frac{du}{d\theta} = \frac{A_n}{gC} u^{n+1} \cos^{n+1} \varphi \sec^{n+1} \theta$$

or

$$\frac{du}{u^{n+1}} = \frac{A_n}{gC} \cdot \text{cas}^n \varphi \,.\, \sec^{n+1} \theta \, d\theta$$

whence, by integration between V, φ, and u, θ,

$$\frac{g\,C}{n\,A_n\,u^n \cos^n \varphi} + (\theta)_n = \frac{g\,C}{n\,A_n\,V^n \cos^n \varphi} + (\varphi)_n = \text{constant} = (\iota)_n$$

and the substitution of the value of u in terms of θ and ι enables the problem to be solved by quadratures.

Thus the expressions

$$dt = -\frac{u \cos \varphi}{g} d \tan \theta$$

$$dx = u \cos \varphi \, dt$$

$$dy = u \cos \varphi \tan \theta \, dt$$

$$ds = u \cos \varphi \sec \theta \, dt$$

become on representing $\frac{g\,C}{n\,A_n}$ by κ^n (which causes the equation of the hodograph to assume the form

$$\left(\frac{\kappa}{u \cos \varphi}\right)^n = (\iota)_n - (\theta)_n)$$

$$dt = -\frac{k}{g} \cdot \frac{d \tan \theta}{((\iota)_n - (\theta)_n)^{1/n}}$$

$$dx = -\frac{k^2}{g} \cdot \frac{d \tan \theta}{((\iota)_n - (\theta)_n)^{2/n}}$$

$$dy = -\frac{k^2}{g} \cdot \frac{\tan \theta \, d \tan \theta}{((\iota)_n - (\theta)_n)^{2/n}}$$

$$ds = -\frac{k^2}{g} \cdot \frac{\sec \theta \, d \tan \theta}{((\iota)_n - (\theta)_n)^{2/n}}$$

By the use of these formulas and by taking small increments of θ so that a mean value of θ will apply in the incremental expression, successive increments may be computed and added. The summation of $\triangle y$ between φ and zero gives the maximum ordinate and $-\omega$ is that value of θ in the descending branch for which the total summation of $\triangle y$ is zero. It is generally found by interpolation between the last two values of θ used.

The Quadratic Law.

This process is best illustrated by the use of the quadratic law of resistance, as that has practical application in mortar fire, and in curved fire.

The equation of the hodograph becomes

$$\frac{k^2}{u^2 \cos^2 \varphi} + (\theta) = \frac{k^2}{V^2 \cos^2 \varphi} + (\varphi) = (\iota)$$

and the expressions for dt, dx, dy and ds become

$$dt = -\frac{k}{g} \cdot \frac{d \tan \theta}{\sqrt{(\iota)-(\theta)}} = -\sqrt{\frac{C}{2\,g\,A_2}} \cdot \frac{d \tan \theta}{\sqrt{(\iota)-(\theta)}}$$

$$dx = -\frac{k^2}{g} \cdot \frac{d \tan \theta}{(\iota)-(\theta)} = -\frac{C}{2\,A_2} \cdot \frac{\tan \theta \; d \tan \theta}{(\iota)-(\theta)}$$

$$dy = -\frac{k^2}{g} \cdot \frac{\tan \theta \; d \tan \theta}{(\iota)-(\theta)} = -\frac{C}{2\,A_2} \cdot \frac{\tan \theta \; d \tan \theta}{(\iota)-(\theta)}$$

$$ds = -\frac{k^2}{g} \cdot \frac{\sec \theta \; d \tan \theta}{(\iota)-(\theta)} = -\frac{C}{2\,A_2} \cdot \frac{\sec \theta \; d \tan \theta}{(\iota)-(\theta)}$$

which may be put into the forms

$$\sqrt{2\,g\,A_2} \cdot \frac{dt}{\sqrt{C}} = -\frac{d \tan \theta}{\sqrt{(\iota)-(\theta)}} = dt'$$

$$2\,A_2 \cdot \frac{dx}{C} = -\frac{d \tan \theta}{(\iota)-(\theta)} = dx'$$

$$2\,A_2 \cdot \frac{dy}{C} = -\frac{\tan \theta \; d \tan \theta}{(\iota)-(\theta)} = dy'$$

$$2\,A_2 \cdot \frac{ds}{C} = -\frac{\sec \theta \; d \tan \theta}{(\iota)-(\theta)} = ds'$$

It is thus clear that dt′, dx′, dy′, ds′, are dependent only on ι and θ and that ι is known when $\frac{V}{\sqrt{C}}$ and φ are known, since,

$$(\iota) = \frac{g\,C}{2\,A_2\,V^2 \cos^2 \varphi} + (\varphi) = \frac{g}{2\,A_2} \cdot \frac{\sec^2 \varphi}{\left(\frac{V}{\sqrt{C}}\right)^2} + (\varphi).$$

It is, then, possible for a given $\frac{V}{\sqrt{C}}$ and φ and θ to calculate and tabulate t′, x′, y′ and s′ from which $\frac{t}{\sqrt{C}}$, $\frac{x}{C}$, etc., may be calculated when A and g are known.

Having calculated t′, x′, s′ etc., for the point of fall we may produce ratios eliminating C, g and A_2 from all except the ratio $\frac{V}{\sqrt{C}}$. Thus, for the point of fall,

$$\frac{T^2}{C} = \frac{T'^2}{2\,g\,A_2}$$

$$\frac{X}{C} = \frac{X'}{2\,A_2}$$

$$\frac{S}{C} = \frac{S'}{2\,A_2}$$

and these being divided by $\frac{V^2}{C}$ give the values of

$$\frac{T^2}{V^2},\ \frac{X}{V^2},\ \frac{S}{V^2},\ \text{etc.}$$

The maximum ordinate is found by summing dy′ from φ to 0 so that

$$\frac{y_0}{C} = \frac{y_0'}{2\,A_2}$$

and this may be divided by $\frac{V^2}{C}$, or $\frac{X}{C}$ and $\frac{y_0}{V^2}$ or $\frac{y_0}{X}$ tabulated.

The velocity of fall is found thus:

The equation of the hodograph gives

$$\frac{\kappa^2}{u^2_\omega \cos^2\varphi} = (\iota) + (\omega)$$

$$\frac{\kappa^2}{V^2 \cos^2\varphi} = (\iota) - (\varphi)$$

Hence

$$\frac{u^2_\omega}{V^2} = \frac{(\iota) - (\varphi)}{(\iota) + (\omega)} = \frac{v^2_\omega \cos^2\omega \sec^2\varphi}{V^2}$$

That is

$$\frac{v_\omega}{V} = \cos\varphi \sec\omega \sqrt{\frac{(\iota) - (\varphi)}{(\iota) + (\omega)}}$$

It is also true that ds′ is directly integrable, thus

$$ds' = -\frac{\sec^3\theta\, d\theta}{(\iota) - (\theta)} = -\frac{d(\theta)}{(\iota) - (\theta)}$$

Hence

$$s' = \int_{\varphi}^{\theta} \frac{-d(\theta)}{(\iota) - (\theta)} = \log_e \frac{(\iota) - (\theta)}{(\iota) - (\varphi)}$$

and this gives a means of readily performing the calculation by quadratures, as follows:

1. Beginning with any desired $\frac{V}{\sqrt{C}}$ and φ, calculate (ι).

2. Calculate $\Delta s'$ for 1° intervals from φ to 0° and then from 0° to beyond $-\varphi$.

3. Designating the successive values of θ used as θ_1, θ_2 etc. take a mean value θ' of θ as $\frac{\theta_p + \theta_{p+1}}{2}$ for any arc, as the pth, and calculate as follows:

$$\Delta y' = \sin \theta' \Delta s$$

The summation from φ to 0° of $\Delta y'$ gives y_0', and the summation should continue until $y' = 0$ in the descending branch, at which point is the point of fall and $\theta = -\omega$.

4. $$\Delta x' = \cos \theta' \, \Delta s$$

Sum from φ to $-\omega$, for X'.

5. The values of v at each point are found from the equation of the hodograph thus

$$v_\theta = V \cos \varphi \sec \theta \sqrt{\frac{(\iota) - (\varphi)}{(\iota) - (\theta)}}$$

and a mean v over the arc is taken as the half sum so that

$$v' = \frac{v_p + v_{p+1}}{2}$$

6. $$\Delta t' = \frac{\Delta s'}{v'}$$

Sum between φ and $-\omega$.

In the Table of High Angle Fire Ratios given in Part I, the quantities given are

$$\frac{V}{\sqrt{C}}, \quad \frac{10\,X}{V^2}, \quad \frac{10\,T}{V}, \quad \omega, \quad \frac{V_\omega}{V}, \quad \frac{y_0}{X} \text{ and } \log \frac{h}{X}$$

$\frac{h}{X}$ is found by multiplying $\frac{y_0}{X}$ by $\frac{h}{y_0}$ as determined by using, in place of φ, a mean of φ and ω, in the expression already deduced for a mean height.

APPENDIX I.

The Altitude Factor.

Consider two points, one of them being h feet above the other. Let δ, p, and t represent the density, pressure and temperature at the height h and δ_0, p_0, t_0, those at the height zero, or at the lower point.

Then, for equilibrium

$$dp = -\delta g\, dh \qquad (1)$$

and for a gas

$$p = Hag\,\delta\, T \qquad (2)$$

in which T is the absolute temperature Fahr. corresponding to height h, or $T = 461+t$, H the height of the homogeneous atmosphere at 0° Fahr., and $a = \frac{1}{461}$. Hence, since $H = 24600$, we have $H a = 53\frac{1}{3}$.

From equation (2)

$$\frac{dp}{p} = \frac{d\delta}{\delta} + \frac{dT}{T} \qquad (3)$$

and from equation (1)

$$dp = -\delta g\, dh \qquad (4)$$

Hence, from (2) and (4).

$$\frac{dp}{p} = -\frac{dh}{H a T}$$

and, from (3),

$$\frac{d\delta}{\delta} = -\frac{dT}{T} - \frac{dh}{H a T}$$

Next, assuming a drop of temperature of b° per foot of height ($b = \frac{1}{300}$ as found by observations in this country and abroad), we have

$$dT = -b\, dh$$

and by integration between T, and T_0, y and 0,

$$T - T_0 = -bh$$

or

$$T = T_0 - bh$$

Hence

$$\frac{d\delta}{\delta} = \left(\frac{1}{H a b} - 1\right) \times \frac{d\,(T_0 - bh)}{T_0 - bh}$$

and integrating between $h = 0$ and $h = h$.

$$\log_e \frac{\delta_0}{\delta} = \left(\frac{1}{H a b} - 1\right) \log_e \frac{T_0}{T_0 - bh}$$

whence

$$\frac{\delta_0}{\delta} = f_a = \left(1 - \frac{bh}{T_0}\right)^{\frac{-1}{H a b} + 1}$$

Using, $T_0 = 520°$ Fahr., corresponding to $t_0 = 59°$ Fahr.

$$H\,a = 53\tfrac{1}{3}$$

$$b = \tfrac{1}{300}$$

this becomes

$$f_a = \left(1 - \frac{h}{156000}\right)^{-\frac{37}{8}}$$

$$= \left(1 - .0064 \,.\, \frac{h}{1000}\right)^{-4.625}$$

The value of h for direct fire is $\frac{2}{3}\,y_0$ and for high angle fire is a function of the elevation and range as given in Part II, under the discussion of the trajectory for which $C = \infty$, h as used in both these cases being the mean height of the trajectory.

For a mean height 7500 feet the value of f_a may be represented by $f_a = \dfrac{1}{1 - .000027h}$, which form holds for high angle fire with sufficient accuracy.

APPENDIX II.

The Coefficient of Form as Determined by the Shape of the Head of the Projectile.

Experiments conducted by the military authorities in this country and in France have furnished sufficient data to clearly indicate that the coefficient of form is proportional to the mean sine of the angle made by the tangent to the ogive with the axis of the projectile. This mean value is taken over the ogival surface.

Let A B C,* in the figure represent half of the longitudinal section of the ogival surface. D is its center. D A = n calibers is its radius. A O = ½ caliber. O Y and O X are the coördinate axes. Let B be any point on the circle A B C, which curve by revolution around the axis O X generates the ogival surface. Let the angle α = G B F represent the inclination of the differential surface d S to the axis O X.

The equation of the ogival section is, then

$$(y + n - \tfrac{1}{2})^2 + x^2 = n^2$$

and hence

$$\frac{dy}{dx} = -\frac{2x}{2y + 2n - 1} = -\tan\alpha$$

Taking the general case of a surface of revolution the generating curve being not necessarily a circle we note that $\sin\alpha$ obtains over an elementary surface $2\pi y ds = 2\pi y \sec\alpha dx = dS$. The mean value of $\sin\alpha$ over this surface is

$$\sin\alpha_m = \frac{\int_0^S \sin\alpha\, dS}{S} = \frac{\int_0^x 2\pi y \tan\alpha\, dx}{S} = \frac{\int_0^y 2\pi y\, dy}{S}$$

For a complete head the y limits from bourrelet to point are ½ and 0.

Hence, for a complete head

$$\sin\alpha = \frac{\pi y^2\Big]_0^{1/2}}{S} = \frac{\pi}{4S}$$

But c = 1 when the head is a complete ogival struck with a radius 2; and when c is desired for any other head we divide $\sin\alpha$ by $\sin\alpha_2$.

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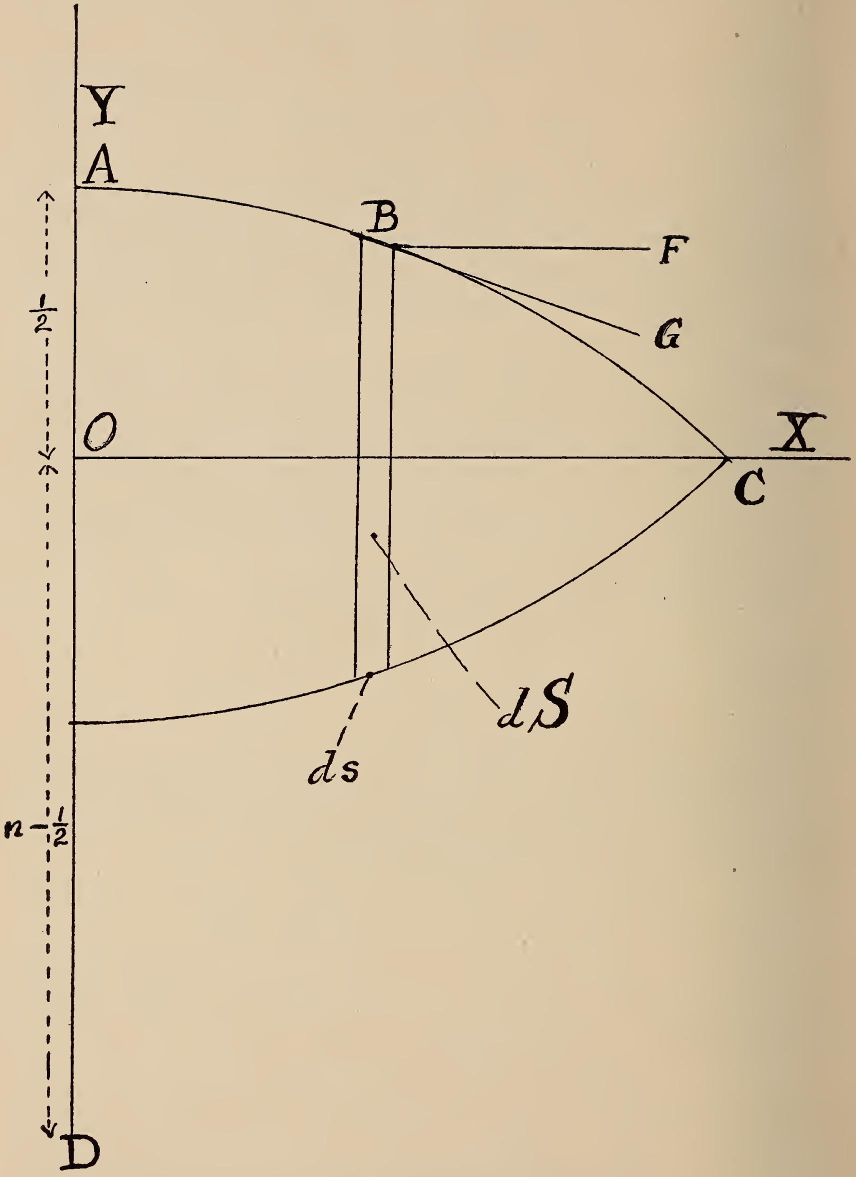
* See illustration on next page.


Y
A
B
F
G
$\frac{1}{2}$
O
X
C
dS
ds
$n-\frac{1}{2}$
D

Thus

$$c = \frac{\sin \alpha}{\sin \alpha_2} = \frac{S_2}{S}$$

That is for a *continuous generating curve* concave toward the x axis at all points. the coefficient of form is inversely proproportional to the surface of the head.

PARTICULAR FORMS OF HEAD.

1. *An Ogive Struck with a Radius n Calibers.*

In this case

$$y = \sqrt{n^2 - x^2} - \frac{2n - 1}{2}$$

$$ds = \frac{n\,dx}{\sqrt{n^2 - x^2}}$$

$$dS = 2\pi y\, ds$$

Hence between $x = 0$ and $x = x_1$

$$S = \int_0^{x_1} 2\pi n \left(dx - \frac{2n-1}{2} \cdot \frac{dx}{\sqrt{n^2 - x^2}} \right)$$

$$= 2\pi n \left(x_1 - \frac{2n-1}{2} \cdot \sin^{-1} \frac{x_1}{n} \right.$$

Now for a complete head x_1 is found by placing $y = 0$, when

$$x_1 = \tfrac{1}{2} \sqrt{4n - 1}$$

Consequently, for a complete ogival head, n-caliber radius,

$$S_n = \pi n \left(\sqrt{4n - 1} - (2n - 1) \sin^{-1} \frac{\sqrt{4n - 1}}{2n} \right)$$

$$= \pi n \left(\sqrt{4n - 1} - (2n - 1)\, A \right)$$

in which A is the angle (in radians) whose sine is $\frac{\sqrt{4n - 1}}{2n}$

For $n = 2$, we have

$$S_2 = 2\pi \left(\sqrt{7} - 3 \sin^{-1} \frac{\sqrt{7}}{4} \right)$$

The values of $\frac{S_2}{S_n}$, or c, for heads with radii from 2 to 7 are given below:

n	c	n	c
2	1.00	5	0.64
3	0.82	6	0.58
4	0.71	7	0.54

These agree with the most recent experiments at Indian Head and at Sandy Hook.

2. *Ogival Heads with Snubbed Points.*

Where the point of a projectile is cut off by a plane perpendicular to its axis, it is said to have a snubbed point.

Let m be the number of inches cut off from the length of the projectile and represent this length in calibers; that is $\frac{m}{d}$, by m'. Then the mean sine from the base of the head to the point x_1, y_1, *i. e.*, to the flat nose, is

$$\sin \alpha = \frac{\pi \left(\frac{1}{4} - y_1^2\right)}{\pi\, n \left(2\, x_1 - (2\, n - 1) \sin^{-1} \frac{x_1}{n}\right.}$$

and for the flat face

$$\sin \alpha = 1$$

The first of these obtains over an aunulus of the cross section of the projectile whose area is $\pi \left(\frac{1}{4} - y_1^2\right)$, and the second over the area of the flat face; which area is $\pi\, y_1^2$.

Multiplying each by the area over which it obtains and dividing by the sum of the areas which is the area of cross-section, we find

$$\sin \alpha = \frac{\pi^2 \left(\frac{1}{4} - y_1^2\right)^2}{\frac{\pi^2 n}{4} \left(2\, x_1 - (2\, n - 1) \sin^{-1} \frac{x_1}{n}\right)} + \frac{\pi\, y_1^2}{\frac{\pi}{4}}$$

$$= \frac{1}{4}\, \frac{\left(1 - 4\, y_1^2\right)^2}{n \left(2\, x_1 - (2\, n - 1) \sin^{-1} \frac{x_1}{n}\right)} + 4\, y_1^2$$

In this equation,

$$x_1 = \tfrac{1}{2} \sqrt{4\, n - 1} - m'$$

$$y_1 = \sqrt{n^2 - x_1^2} - \frac{2\, n - 1}{2}$$

Hence for a snubbed point

$$c = \frac{\sin \alpha}{\sin \alpha_2} = 3.824 \sin \alpha$$

Representing $\sin^{-1} \frac{x_1}{n}$ by A

$n\, (2\, n - 1)\, A$ by A'
$n\, (2\, x_1 - (2\, n - 1)\, A)$ by $B = 2\, n\, x_1 - A'$
$\frac{1}{4}\, (1 - 4\, y_1^2)^2$ by D

we have

$$c = 3.824 \left(\frac{D}{B} + 4\, y_1^2\right)$$

Ex. A 6 inch projectile whose head is struck with a radius of 7 calibers has its point snubbed (a) $2\frac{1}{4}''$; (b) $1\frac{1}{2}''$; (c) $\frac{3}{4}''$. Find c in each case.

Here

	(a)	(b)	(c)
m	2.25	1.50	0.75
m′	0.375	0.250	0.125
x_1	2.22308	2.34808	2.47308
$2\,n\,x_1$	31.12312	32.87312	34.62312
A	.32318	.34207	.36108
A′	29.40938	31.12837	32.85828
y_1	.1376	.0946	.0486
D	.21356	.23242	.24530
B	1.71374	1.74475	1.76484
$\frac{D}{B}$	.12405	.13321	.13899
$4\,y_1^2$	.07574	.03580	.00945
sin α	.19979	.16901	.14844
c	.764	.646	.568

The general formula

$$c = \frac{S_2}{S}$$

can be applied to any complete head of parabolic or other form provided its curvature is continuous and concave toward the axis of the projectile. Snubbing can be dealt with in a manner analogous to that used in the case of ogival projectiles.

The formula given in Part I, viz.:

$$c = \frac{2}{n}\sqrt{\frac{4\,n - 1}{7}}$$

assumes that c is proportional to the value of α *at the* point. This may be shown as follows:

$$\tan\alpha = \frac{2\,x}{2\,y + 2\,n - 1}\;;\quad \sec\alpha = \frac{n}{\sqrt{n^2 - x^2}}$$

and y = 0 at the point, and $x = \frac{1}{2}\sqrt{4\,n - 1}$

Hence at the point

$$\tan\alpha = \frac{\sqrt{4\,n - 1}}{2\,n - 1}\;;\quad \sec\alpha = \frac{2\,n}{(2\,n - 1)}$$

whence

$$\sin\alpha = \frac{\tan\alpha}{\sec\alpha} = \frac{\sqrt{4n - 1}}{2\,n}$$

Hence

$$\sin \alpha_2 = \frac{\sqrt{7}}{4}$$

and

$$c = \frac{\sin \alpha}{\sin \alpha_2} = \frac{4}{2n}\sqrt{\frac{4n-1}{7}} = \frac{2}{n}\sqrt{\frac{4n-1}{7}}$$

This formula gives sensibly the same results as the formula

$$c = \frac{S_2}{S}$$

but is not applicable except to ogival heads which are complete.

Conical Snubbed Points.

These are such that the point is not cut off flat but a short cone used instead. The inclination of the elements of this cone will lie between 90° and the point inclination of the complete head.

As already shown, $\sin \alpha$ for the complete head is given by the equation

$$\sin \alpha = \frac{\sqrt{4n-1}}{2n}$$

and $\sin 90° = 1$.

Taking the inclination of the elements of the short cone we may find c by interpolation between the value of c found for the complete head and that for a snubbed point.

APPENDIX III.

Experimental Determinations of F (v).

The value of F(v) has been determined with considerable accuracy in recent years.

The values used in our present ballistic tables are from the Krupp firings at Meppen and the formulas through which they are applied are due primarily to Mayevski and finally to Zabudski who extended the formulas to values of v as high as 3600 f.s.

From current firings it is, and has been for some years, evident to the writer that these laws are only approximately correct; and upon a careful examination of the firings conducted by the Gâvre Commission the respects in which they are inaccurate are clearly shown.

At the same time, certain interesting relations appear from the firings of the Gâvre Commission. It appears that Newton's Law of the Square is, in the mean, true; since if we take the velocity 330 metres per second (1083 f.s.) and its experimental F(v), as determined by the Gâvre Commission, it admits of the determination of a parabolic relation $y = Kv^2$ giving a parabola asymptotic to the actual curve $F(v) = y$.

This is made plain by plotting the curve $\frac{F(v)}{v^2} = y'$ and the straight $y' = K$. The latter is clearly the asymptote of the former.

The law of air resistance seems to be intimately connected with the velocity of sound. As given by the firings of the Gâvre Commission it almost obeys the law.

$$\frac{F(v)}{v^2} = a + \frac{b\,(v - v_s)}{c^2 + (v - v_s)^2}$$

in which v_s is the velocity of sound in air.

Inasmuch as the Krupp experiments do not agree with the Gâvre results altogether and the latter are not capable of complete expression by the equation connected with the velocity of sound, it is considered advisable to reproduce the Gâvre laws otherwise.

Representing by K (v) the ratio $\frac{F(v)}{v^2}$, it is noted from the experiments that $K(v - 1083) + K(v + 1083) = 2K(1083)$. That is, the curve is central with respect to the velocity of sound 1083, which is the value indicated by the firings. The general mean velocity of sound deduced by Rowland is 331.78 metres per second at 0° C, or 1089 f.s., in pure dry air.

The minimum value of K(v) occurs when v = 330—155 m.s. or v = 175 m.s. = 574 f.s., and the maximum value for v = 330 + 155 m.s. = 485 m.s. = 1591 f.s.; the mean of which is 1082.5 f.s. The corresponding values of K(v) are $\frac{4.13}{10^4}$ and $\frac{13.61}{10^4}$ giving a mean of $\frac{8.87}{10^4}$. The value $\frac{8.87}{10^4}$ is given for 1083 f.s. by the experiments. These facts suffice to determine the curve, if it is of the form outlined above, since in that case

$a = .0000887$
$b = .0000933c$
$c = 508$

This determination gives a curve astonishingly like the experimental curve and leads to the supposition that some curve of that form must represent the law. Maxima and minima are hard to locate on an experimental curve and the probable error of the determinations is sufficiently great to render it possible to find a curve of the above form giving correct results.

The probable cause of the errors of determination which undoubtedly exist, is the failure to record the exact condition of the atmosphere with respect to the ratio of the specific heats, which exercises a marked influence on the velocity of sound. For this reason the result is more or less composite in the vicinity of the velocity of sound, which in ordinary air at mean temperature and humidity is higher than 1089 f.s.

In our present tables, given in Artillery Circular M, the laws used were those of Zabudski, based on the Krupp firings. They are of the form

$$F(v) = A_n v^n$$

over certain velocity-ranges, and rendered compatible by the adjustment of A when n is changed. In integrating, to find the S, A, I, and T functions, constants of integration are of course necessary to ensure compatibility in the determination of these functions. (See Appendix, Artillery Circular N.) These velocity ranges are sometimes quite extended and the mean value A assumed as a coefficient of v^n departs materially from many of its variable values; aside from this, the determinations are not the most recent ones.

It is thought best to give the value of K(v) as found from the firings of the Gâvre Commission and quoted in full by Charbonnier

(*Traite de Balistique Extérieure,* 1905). The use of these values in computing the ballistic functions will appear.

In Table I the values of v are taken at such intervals that ordinary or linear interpolation only is required between tabular values.

TABLE I.

Values of $10^4 K(v) = \frac{F(v)}{v^2} \times 10^4$

v f. s.	$10^4 \times K(v)$	v f. s.	$10^4 \times K(v)$
0	0.5385	1214	1.1784
263	.4689	1247	1.2246
492	.4202	1279	1.2581
558	.4136	1312	1.2862
590	.4133	1345	1.3071
623	.4147	1378	1.3213
656	.4172	1411	1.3357
689	.4206	1443	1.3452
722	.4276	1476	1.3524
787	.4484	1509	1.3558
820	.4658	1542	1.3580
853	.4866	1575	1.3594
886	.5149	1607	1.3561
919	.5481	1673	1.3527
951	.5902	1903	1.3038
984	.6465	2297	1.1995
1017	.7168	2526	1.1505
1050	.7967	2920	1.0879
1083	.8863	3150	1.0637
1115	.9761	3510	1.0445
1148	1.0561	3609	1.0411
1181	1.1262	3937	1.0325
1214	1.1784	4000	1.0308

Table I gives the means of interpolating so as to find K(v) for each f.s. from 0 to 4000 f.s. Multiplying in each case by v^2 we obtain F(v). We may thus readily compute by simple quadratures the S, A, I, and T functions, adding increments from one velocity to the next at intervals of 1 f.s., or more.

To do this beginning with 4000 f. s. the value of K(v) for which is 10.308, we have

$$d\,S\,(u) = -\frac{u\,d\,u}{F\,(u)}$$

$$d\,I\,(u) = -\frac{2\,g\,d\,u}{u\,F\,(u)}$$

$$d\,T\,(u) = -\frac{d\,u}{F\,(u)}$$

$$d\,A\,(u) = I\,(u)\,d\,S\,(u).$$

The S and T functions may be made equal to zero at 4000 f. s.; also the A function. The initial value of the I function may be determined as follows:

Represent K (v) by K, for simplicity.

Placing $F\,(u) = K\,u^2 = \frac{1.0308}{10^4}\,u^2$, we have

$$d\,I\,(u) = -\frac{2\,g\,d\,u}{u\,F\,u} = -\frac{2\,g}{K}\,\frac{du}{u^3}$$

Hence

$$I\,(u) = \frac{2\,g}{K} \times \frac{1}{2\,u^2} = \frac{g}{K\,u^2} + \text{constant}$$

$$= \frac{g}{K\,u^2} + k_i$$

Now

$$d\,A\,(u) = I\,u\,d\,S\,u$$

$$= \left(\frac{g}{K\,u^2} + k_i\right)\left(-\frac{u\,d\,u}{K\,u^2}\right)$$

$$= -\frac{g\,d\,u}{K^2 u^3} - \frac{k_i}{K} \cdot \frac{du}{u}$$

Hence

$$A\,(u) = \frac{g}{2\,K^2\,u^2} - \frac{k_i}{K}\,\log_e u + k_a$$

If now k_i be made zero and $k_a = -\frac{g}{2\,K^2 (4000)^2}$ it follows that k_i being zero

$$I\,(4000) = \frac{g}{K\,(4000)^2} = \frac{g}{F\,(4000)}$$

The initial values having been determined the solution is one of ordinary computation by quadratures.

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